17/1 (2019), 93–103
DOI: 10.5485/TMCS.2019.0462
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Teaching Mathematics and Computer Science

Balanced areas in quadrilaterals – Anne's Theorem and its unknown origin

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Abstract. There are elegant and short ways to prove Anne's Theorem using analytical geometry. We found also geometrical proofs for one direction of the theorem. We do not know, how Anne came to his theorem and how he proved it (probably not analytically), it would be interesting to know. We give a geometric proof (both directions), mention some possibilities – in more details described in another paper – for using this topic in teaching situations, and mention some phenomena and theorems closely related to Anne's Theorem.

Key words and phrases: Geometry, History of Mathematics.

ZDM Subject Classification: G10, G30.

Introduction

When generalizing a problem of elementary geometry some interesting questions appeared which led us – this emerged by investigations after finishing our work – to the so called Anne's Theorem. This is a fairly unknown theorem concerning convex quadrilaterals which seems to go back to the French mathematician Pierre-Leon Anne (1806 – 1850). We neither know how Anne has proven "his" theorem at his time nor how he discovered it[1], but the proof in Alsina, Nelsen (2010, p. 116f), Honsberger (1991, p. 174f) is not a proof using arguments of elementary

[1]Our investigations to this topic (also asking competent experts) were not successful. None of the asked colleagues even knew the theorem – also we did not know it up to a short time ago.

geometry, and we assume that this idea was not the "original" one of Anne. Following Honsberger the idea of this proof goes back to the Australian mathematician Basil Rennie (1920 – 1996), it uses an argument of linearity in a very smart way involving "calculations with coordinates" (analytical geometry). In several publications that use geometry for dealing with the theorem only one direction of Anne's Theorem is proven, the easier one (Jobbings (2013), or https://www.cut-the-knot.org/Curriculum/Geometry/NewtonTheorem.shtml). We did not find references which use only elementary geometry and treat both directions of Anne's Theorem. Our proof is of that kind, and maybe also Anne's thoughts were somehow similar? It would be very interesting for us to know. There is also a theorem called Newton's Theorem (concerning quadrilaterals having an incircle) which is a direct corollary of Anne's Theorem. But also in this case we have no idea how Newton came to this theorem and how he has proven it. Was this theorem really discovered and proven by Newton? Or did somebody else (many years after Newton) ascribe this theorem to Newton? How had it been proven before the times of Anne's theorem (in case it existed already)? Many interesting questions in the field of the history of mathematics arise. We don't know the answers but we would be highly interested in them. Due to this not clear connection to Newton a special straight line that plays a very important role in the following sections is also called "Newton line" (up to a short time ago also this "name" was unknown to us).

The initial problem

Problem: In a square an arbitrary point I is connected with the vertices, the resulting triangles are colored alternately gray and white (opposite triangles have the same color). Then the area sum of the gray triangles equals the area sum of the white ones (Figure 1).

Here on the one hand DGS (Dynamic Geometry Software) experiments are easy to do and on the other hand the corresponding proof is easy to give. Very similar is the question if one takes parallelograms instead of squares (first step of generalizing). In a second step of generalizing one can examine the situation when taking trapezoids or kites instead. The result will be: Here the point I cannot be moved arbitrarily in the interior if it should lead to balanced areas. But at least the points of some special lines seem to have the area balance property (the midsegment of the trapezoid, the diagonal of symmetry of a kite). For details

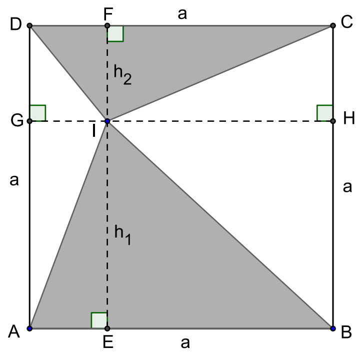

Figure 1. Square

concerning this topic in possible teaching situations (mathematics as a process, problem solving, etc.) see Humenberger (2018).

These results fit well to Anne's Theorem which deals with that problem in the general case of convex quadrilaterals.

Anne's Theorem

Anne's Theorem: Let $ABCD$ be a convex quadrilateral which is not a parallelogram. Then all points I with the area balance property (between the gray and white triangles) lie on the line $g = MN$ through the midpoints of the two diagonals.

One can see immediately that M and N have this property. (But: If these two special points were not mentioned in the theorem it would be not so easy to discover their special role in this problem; one would have to use heuristic strategies.) Exploring the situation with DGS: One can construct with DGS the straight line $g := MN$ and fix the point I to g (see Figure 2); then one can observe that the white and gray triangles have exactly equal area sums; if one "releases" I from g then the result will be: for $I \notin g$ the area sums are different. This is an impressive experimental confirmation of the theorem by using DGS, something impossible up to some 20 years ago. Throughout this article we restrict ourselves to *interior* points of the quadrilateral.

Proof: At first we prove the following statement by means of elementary geometry: When moving the point I on g then the area sums of the gray and white triangles do not change. As we know that the points M, N have the area

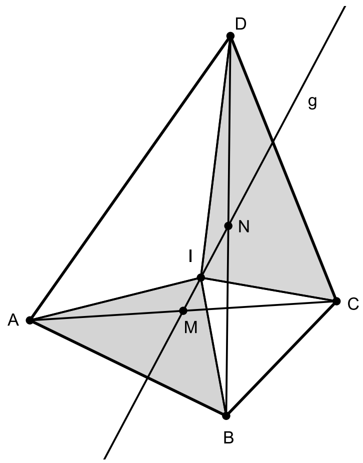

Figure 2. The line $g = MN$

balance property we then will know that all points of g have this property, too. Let I, K be two different points of g; we will show first that the two "arrows" $AIBK$ and $CIDK$ (see Figure 3: dotted and dashed) have the same area. Because M lies on g the points A and C have the same distance h_1 from g; analogously B, D have the same distance h_2 from g because also N lies on g. Both arrows consist of two triangles with the same side IK; one of the triangles has the altitude h_1 the other one the altitude h_2. Therefore, both arrows have the same area $\frac{1}{2} \cdot |IK| \cdot (h_1 + h_2)$.

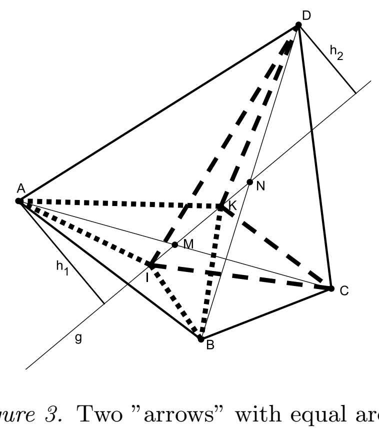

Figure 3. Two "arrows" with equal area

When we move point I to K then we can establish the following considerations of "increasing and decreasing" for the gray and white areas (see Figure 4a before the motion, 4b afterwards; the color of triangles like ΔARD or ΔAIB does not change):

(1) The quadrilateral $IRKS$ (overlapping of the arrows) was gray and stays gray.

(2) The triangles RKD and SKC change their color from gray to white.

(3) The triangles RIA and SIB change their color from white to gray.

Because the arrows have equal areas also the area sums of the remaining triangles in (2) and (3) resp. are equal.

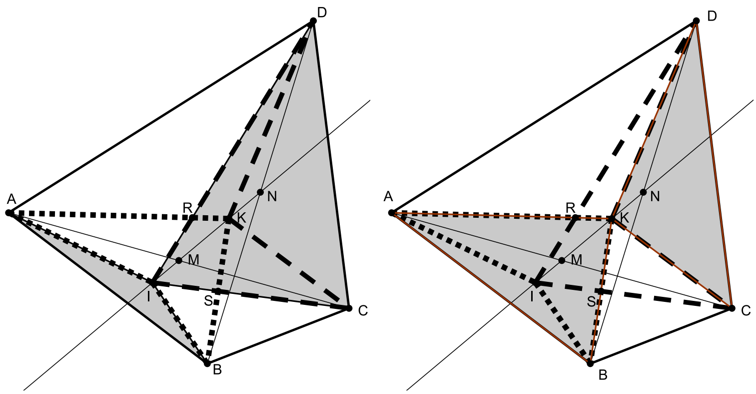

Figure 4. The gray area when moving the point I to K: before and afterwards

For this first part of the proof one can find references that deal with the same question (Jobbings (2013) or https://www.cut-the-knot.org/Curriculum/Geometry/NewtonTheorem.shtml). But it remains to check: What happens for points $I \notin g$?

As said above the area sums can easily be calculated if I moves on a diagonal; this we can use – mutatis mutandis – also here. For $I = M$ the area sums are equal; moving I away from M on the diagonal AC the area sums will usually change (see Figure 5). Because the triangle MID changes to gray, the triangle MIB changes to white; the triangles usually have different areas because the altitudes from B and D to MI and AC are different (except it happens that N lies on AC; in this case we would take the other diagonal BD for the described operation).

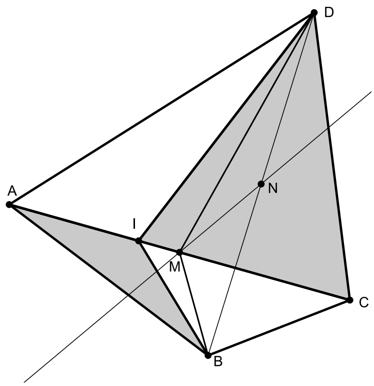

Figure 5. Moving the point I away from M on the diagonal AC

The rest of the argumentation relies on an observation which is based on a more general question: For which points I is the area sum of equally colored triangles constant (not necessarily the half of the quadrilateral area)? The following conjecture seems likely (it can easily be confirmed by DGS experiments): This is the case when I moves on a line p which is *parallel* to g. The above proof for $I \in g$ holds also in this case if one considers the following (see Figure 3; one has to interpret the figure dynamically): When moving the diagonal IK of an arrow ("base side" of "its" triangles) on a line which is parallel to g then the altitude of the one triangle will increase the altitude of the other triangle will decrease by the same amount. Thus, the sum of the altitudes and the area of the arrow will not change. Both arrows (the not moved and the moved one) have the same area $\frac{1}{2} \cdot |IK| \cdot (h_1 + h_2)$.

Result: If I does not lie on g then move I parallel to g onto a diagonal (the area sums do not change); for I on a diagonal (say AC) we know (see above, Figure 5): the area sums for $I = M$ and $I \neq M$ resp. are different! This completes the proof.

Let us formulate the used and proven phenomenon in an own **Theorem** (we did not find any references for it): Let $ABCD$ be a convex quadrilateral which is not a parallelogram and the line $g = MN$ the straight line through the midpoints M, N of the two diagonals. Then the loci of all points I with the property $|\Delta ABI| + |\Delta CDI| =$ constant are the parallels to g.

Other Theorems and phenomena connected to Anne's Theorem

We would like to mention further observations from the DGS experiments (easy to show if wanted) which we did not use in the proof above:

(1) The intersection points of g and the sides of the quadrilateral divide these sides in the same ratio.

(2) The midpoint of the segment MN is the center of the quadrilateral (parallelogram) built by the midpoints of the given quadrilateral's sides.

(3) It can be an interesting problem of elementary geometry to show in a different way that the midpoint mentioned in (2) has the balanced area property.

In most cases Anne's Theorem is mentioned together with **Newton's Theorem** (see Figure 6): In a quadrilateral that has an incircle the center of the incircle lies on the straight line through the midpoints M, N of the diagonals.

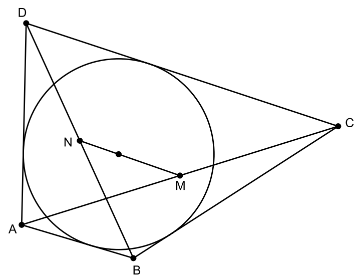

Figure 6. Newton's Theorem

Taking into account the characteristic property of such quadrilaterals that the sum of the lengths of opposite sides is equal, this theorem is a direct corollary of Anne's Theorem. But: How has Newton's Theorem been proven before Anne's Theorem existed? Did it actually exist before? Here interesting questions arise concerning the history of mathematics. But unfortunately, we do not have an answer.

Due to this connection the straight line MN is often called *Newton line*. It would be very interesting to find out in which context Newton came to this theorem because usually such results do not arise without contexts. But such information is hard to get. Even our oldest reference Serret (1855) says nothing about historical connections[2]. Possibly Newton himself never formulated this theorem and somebody else (who? why?) ascribed this theorem to Newton many years (decades? centuries?) later. A small hint is given by the following footnote in Hofmann (1958, p. 200), originally in German, translated by the authors): "The problem should be related to the determination of the locus of all centers of ellipses inscribed a convex quadrilateral. I could not find such considerations in the works of Newton."

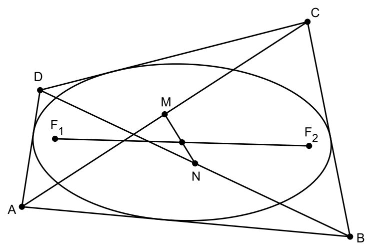

Figure 7. Inscribed ellipse

Indeed: When one inscribes an ellipse into a convex quadrilateral then its center lies on the Newton line (see Figure 7; there are many ellipses inscribed a convex quadrilateral; we do not deal with the question how to construct them).

It is possible to use Newton's Theorem for a proof: "compressing" the quadrilateral in the direction of the major axis (i.e. using a special axial affinity with factor < 1; axis = minor axis) makes a circle out of the ellipse, the result is a quadrilateral with an incircle; the involved properties are preserved in this process, i.e. the center of the ellipse "maps to" the center of the incircle and the compressed Newton line of the quadrilateral is the Newton line of the compressed quadrilateral. It is well possible that Newton reduced the problem of finding the locus of the centers of inscribed ellipses in this way to dealing with quadrilaterals that have an incircle. The "core" of this "solution" then may have become independent in some way as "Newton's Theorem" and this in turn eventually has become a corollary of Anne's Theorem? We can observe this phenomenon in mathematics quite often: When a problem is solved a crucial part of the solution becomes independent of the initial context and "starts a new life"; it can generate

[2] It is also possible that there is no "original paper" of Anne in which he published "his" theorem including a proof. Maybe he had only some notices written by hand or only told orally his "discovery" to some colleagues who published papers to this topic referring in the theorem the name Anne (e.g. Serret (1855))?

new terms and perceptions and even may become the core of a new theory (of course, the famous examples are more substantial than this little one). In such cases it is usually not easy to reconstruct the origins.

The high quality of geometrical problems often corresponds to the amount in which they have close connections to other topics. This applies also to Anne's Theorem as shown by the following further properties of the diagonals' midpoints and the Newton line.

The straight line MN sometimes is also called "Gauss line" because of **Gauss' Theorem**: If the quadrilateral $ABCD$ is no trapezoid then we consider the intersection points E and F of the pairs of opposite sides (straight lines) AB, CD and AD, BC resp.; let O be the midpoint of the line segment EF. Then the points M, N, O lie on a common straight line.

Furthermore we want to mention **Euler's Theorem**: In an arbitrary quadrilateral let $e := |AC|$, $f := |BD|$, $g := |MN|$; then the following relation holds: $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4 \cdot g^2$ (see Figure 8)

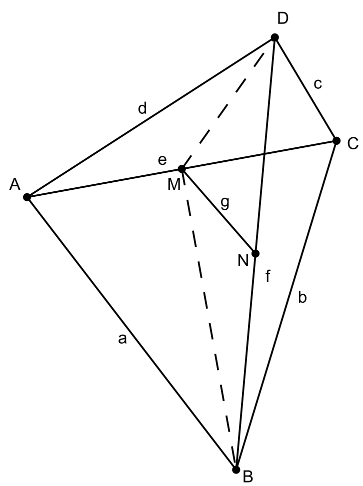

Figure 8. Euler's Theorem

Here we want to give ideas for a possible proof which is charming primarily because of its structure: The general case is reduced by clever calculations to a special case which is easy to prove (please consider missing details by your own).

a) For parallelograms we have $M = N$, therefore $g = 0$; further $a = c$ and $b = d$, and the theorem reduces to $2 \cdot (a^2 + b^2) = e^2 + f^2$; this is easy to prove by using Pythagoras' Theorem.

b) Transferred to a triangle (half a parallelogram) the theorem of a) looks like (in order not to destroy the notations of the quadrilateral we denote the sides of the triangle with x, y, z; the median line to x is denoted by m_x): $y^2 + z^2 = \frac{x^2}{2} + 2 \cdot m_x^2$.

c) For the general quadrilateral we apply b) three times: In ΔABC and ΔACD resp. we have $a^2 + b^2 = \frac{e^2}{2} + 2 \cdot |BM|^2$ and $c^2 + d^2 = \frac{e^2}{2} + 2 \cdot |DM|^2$ resp.; in ΔBDM we get $|BM|^2 + |DM|^2 = \frac{f^2}{2} + 2 \cdot g^2$. Addition of the first two equations and inserting the third one immediately yields the theorem.

Concluding we mention an interesting problem connected to the midpoints of diagonals in a quadrilateral (cf. Simon (1906, p. 156); see Figure 9a; for an older reference cf. Brune (1841)): Draw a parallel line to the diagonal AC through N and a parallel line to BD through M; they intersect in E. Joining E with the midpoints of the sides gives *four sections* with *equal area*!

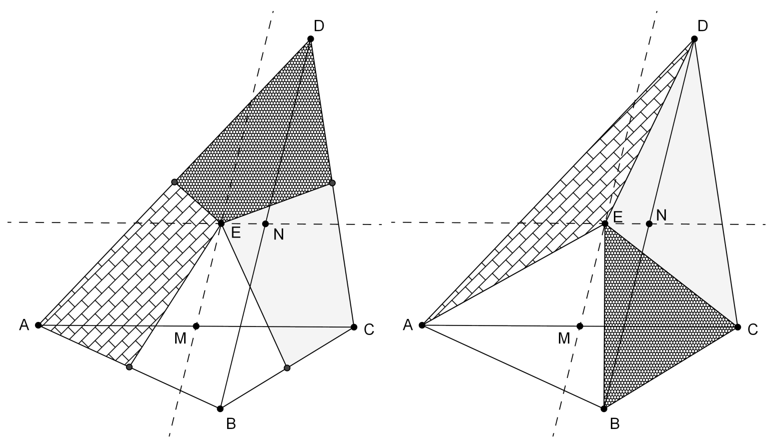

Figure 9. Further interesting problems connected to Anne's Theorem

A variation directly results in another problem: When joining E with the vertices of the quadrilateral then not all parts have same area but still both pairs of opposite triangles do have this property (see Figure 9b). This figure is built up similarly to the initial problem concerning Anne's Theorem but the question is a different one. One could generalize in the following way: For which points I is there a single pair of opposite triangles with equal area? Maybe the result is again a straight line and intersecting the two straight lines corresponding to both possible pairs of opposite triangles one gets the point E?

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(Received February, 2019)