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Trigonometric identities via combinatorics

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Abstract. In this paper we consider the combinatorial approach of the multi-angle formulas $\sin n\theta$ and $\cos n\theta$. We describe a simple "drawing rule" for deriving the formulas immediately. We recall some theoretical background, historical remarks, and show some topics that is connected to this problem, as Chebyshev polynomials, matching polynomials, Lucas polynomial sequences.

Key words and phrases: multiple angle formulas, Chebyshev polynomials, matching polynomials, Fibonacci and Lucas numbers, combinatorial identities.

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Introduction

The funded knowledge of trigonometric functions are crucial in several studies. Moreover, for applications, the deep knowledge of properties, relations and identities are important, since the students should also be able to prove given identities, or eventually to derive new relations. In spite of their importance, the practice shows that students often face difficulties during dealing with trigonometric functions. These functions are somehow foreign for many students and find it hard to manipulate with them. It is a hard didactic question how to make students familiar with trigonometric functions in order to deal with them with facility. Here, in this work we make an attempt into this direction. We would like to add at least one little piece to this serious problem, as we present a simple idea how to "draw" the formulas for the expressions $\sin n\theta$ and $\cos n\theta$, with $n \in \mathbb{N}$.

We hope, such a simple way to obtain the result, domesticates these formulas and helps to memorize them, or at least gives an easy tool how to derive them fast whenever they are needed in calculations.

Another motivation of this paper is to point out the use of combinatorics as a tool, as a help for other areas. Unfortunately, the teaching of combinatorics is mostly isolated, and the students may have the impression that combinatorics has to be learned for its own sake. In the practice - according to the students -, many teachers themselves tend to neglect this discipline. Obviously, combinatorics is also important for applications: the most obvious example is the probability theory which in my opinion can not be understood without having a sense of combinatorial thinking, and is today so fundamental in every science, that it is part of the curriculum in almost all areas, not only in specific mathematical studies.

The example we present should show the students that combinatorial thinking can help, and can make things easier. Moreover, it can be used! Used, also in areas, where we would not expect.

A third point, that motivated to write this article is that the mathematics behind the simple "drawing" is really beautiful, deep mathematics, that connect trigonometry, graph theory, and enumerative combinatorics, which is worth to present for a wider audience.

Finally, we think that this is a beautiful example for concepts, objects that arise in different areas in mathematics showing the wealth of their nature.

We emphasize, that the background should not be told for all the students, but it could wake up the interest, and of course in this case the explanation is expected.

The aim of this article is to wake up the interest, to broaden the view. For this reason, we give an exhaustive description of the connecting mathematical theories and also highlight historical backgrounds and we let the reader to decide for which part how deep he or she is interested.

The outline of the paper is as follows. First, we recall the elementary way to derive the formulas for $\sin n\theta$ and $\cos n\theta$, the way how it is taught in the high school. Next, we present the derivation of the formulas using complex numbers and de Moivre formulas. After all, we present the "drawing" rule. The background of the drawing rule is explained in the next sections: in Section 3 we consider the relations to Chebyshev polynomials, in Section 4 we present the combinatorial approach, revealing the strong connections to matching polynomials. Finally, in Section 5 we describe the connection to the family of polynomials involving Fibonacci polynomials and present a simple example of the use of the combinatorial approach for proving further trigonometric identities.

Multiple-angle formulas

Elementary approach

In high school the calculation of $\sin nx$ and $\cos nx$ is based on the addition formulas

PROPOSITION 1.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

PROOF. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors of length 1:

$$\mathbf{a} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j};$$
$$\mathbf{b} = \cos\beta\mathbf{i} + \sin\beta\mathbf{j},$$

with $\alpha > \beta$. The scalar product of the two vectors can be calculated in two ways: product of the length of the vectors and the cosine of the angle between the vectors and as the sum of the products of the coordinates, respectively.

$$\mathbf{ab} = 1\cdot 1\cdot\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \tag{1}$$

The expression for $\cos(\alpha+\beta)$ follows from (1) by setting $-\beta$ instead of β and from the fact that $\cos x$ is an even function, i.e., $\cos(-\alpha) = \cos\alpha$. Similarly, the addition formula for $\sin(\alpha+\beta)$ follows from (1), by setting $-\beta$ instead of β and $\frac{\pi}{2}-\alpha$ instead of α, using the relation $\sin\alpha = \cos(\frac{\pi}{2}-\alpha)$ and the fact that $\sin x$ is and odd function, i.e., $\sin(-\alpha) = -\sin\alpha$. □

Clearly, setting $\alpha = \beta$ we obtain:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \quad \text{and}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

In order to express $\cos 2\alpha$ as a function of $\cos\alpha$, we set $\sin^2\alpha = 1-\cos^2\alpha$:

$$\cos 2\alpha = 2\cos^2\alpha - 1.$$

Similarly, with $3\alpha = 2\alpha + \alpha$ we obtain for $\cos 3\alpha$:

$$\begin{aligned}\cos 3\alpha &= (\cos^2\alpha - \sin^2\alpha)\cos\alpha - 2\sin^2\alpha\cos\alpha \\ &= \cos^3\alpha - 3\sin^2\alpha\cos\alpha = \cos^3\alpha - 3(1-\cos^2\alpha)\cos\alpha \\ &= 4\cos^3\alpha - 3\cos\alpha.\end{aligned}$$

For $\sin 3\alpha$ we get:

$$\begin{aligned}\sin 3\alpha &= \sin(2\alpha+\alpha) = \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\ &= 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha \\ &= \sin\alpha(3\cos^2\alpha - (1-\cos^2\alpha)) = \sin\alpha(4\cos^2\alpha - 1).\end{aligned}$$

Continuing this way, with some calculations, we obtain for any n formulas for $\cos n\alpha$ and $\sin n\alpha$ and it reveals that $\cos n\alpha$ can be expressed as a polynom of $\cos\alpha$ and $\sin n\alpha$ can be expressed as a product of $\sin\alpha$ and a polynom of $\cos\alpha$. In the next subsection we prove this fact using the de Moivre's formula.

De Moivre's formula

We switch to the in this context more usual notation for the angle θ, instead of α. Recall that the Euler formula states

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{2}$$

where i denotes the imaginarity unit with $i^2 = -1$. The nth power of $e^{i\theta}$ is the de Moivre's formula:

$$(e^{i\theta})^n = e^{i(n\theta)} = \cos n\theta + i\sin n\theta.$$

Otherwise, by the binomial theorem we have

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}(i\sin\theta)^k\cos^{n-k}\theta.$$

(We use the notation $\lfloor x\rfloor$ for the greatest integer that are less than x.) The real parts of the two expressions are equal, hence, we have

$$\cos n\theta = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n}{2k}\cos^{n-2k}\theta\sin^{2k}\theta. \tag{3}$$

We see, that in (3) only even powers of $\sin\theta$ appear, which can be replaced by even powers of $\cos\theta$:

$$\sin^{2k}\theta = (1-\cos^2\theta)^k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\cos^{2j}\theta. \tag{4}$$

After subsitution and reindexing, we obtain

$$\cos n\theta = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^j\left(\sum_{k=j}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}\binom{k}{j}\right)\cos^{n-2j}. \tag{5}$$

shows that $\cos n\theta$ is indeed a polynomial in $\cos\theta$. Similarly to (3),

$$\sin n\theta = \sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}(-1)^{k-1}\binom{n}{2k-1}\cos^{n-(2k-1)}\theta\sin^{2k-1}\theta. \tag{6}$$

Here, we see that the odd powers of $\sin\theta$ appear. We need to separate the factor $\sin\theta$ before the substitution of (4).

$$\sin n\theta = \sin\theta\left(\sum_{k=1}^{\lfloor \frac{n+1}{2}\rfloor}(-1)^{k-1}\binom{n}{2k-1}\cos^{n-2k+1}\left(\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\cos^{2j}\right)\right). \tag{7}$$

The expression (7) implies that $\frac{\sin n\theta}{\sin\theta}$ is also a polynom in $\cos\theta$. In the simpler cases the de Moivre formula leads to a shorter calculation than the elementary approach (using for instance the Pascal triangle for evaluating the binomial theorem).

$$\begin{aligned} e^{2i\theta} &= (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\ e^{3i\theta} &= (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \end{aligned}$$

We have seen that the expressions $\cos n\theta$ and $\frac{\sin n\theta}{\sin\theta}$ are polynomials in $\cos\theta$. These polynomials are the well-known and well-studied polynomials, the Chebyshev polynomials of the first and second kind. Section 3 is devoted to these polynomials.

,,Drawing-rule”

Now we turn our attention to an easy and direct rule, how we can ”derive” the formulas at once, using some simple pictures. The theoretical background of the rule is described in Sections 3, 4, and 5.

First, we consider the expressions for $\sin n\theta$. We take a path of length n and list all matchings of the paths. We obtain a matching of a path if we select some edges (denoted by bold and red in the figure (Figure 1)) such that two selected edges do not share any endnodes. We associate to each isolated vertex (node that is not attached to any selected edge) the weight $2\cos\theta$ and for each selected edge the weight -1. The weight of a path is the product of all the associated weights (of all isolated vertices and selected edges.) Summing over all possibilities and multiplying by $\sin\theta$ we obtain the formula for $\sin(n+1)\theta$.

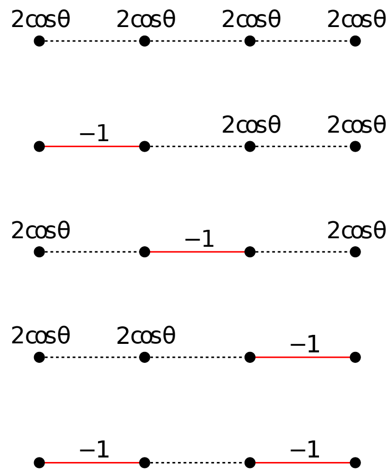

Figure 1. ,,Drawing" $\sin 5\theta$

For instance, for $n = 4$ we have five possibilities, see the following figure (Figure 1). We have three possibilities: we do not select any edge, we select one edge (that can be done three ways), or we select two neighboured edges. In the first case, the associated weight is $(2\cos\theta)^4$, since all the four vertices are isolated. As a next term, we obtain $-3(2\cos\theta)^2$, since one selected edge gives a factor -1, while two isolated vertices give two $2\cos\theta$ factors. Finally, in the case with two selected edges, to each edge -1 is associated, hence, this matching has weight 1. Hence, we have

$$\sin 5\theta = \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1).$$

Our next "drawing rule" determines a formula for $\cos n\theta$ expressed as powers of $\cos\theta$. Take a cycle of length n and list all matchings of the cycle. We associate the weights similarly as in the case of a path. Finally, we sum all the possibilities

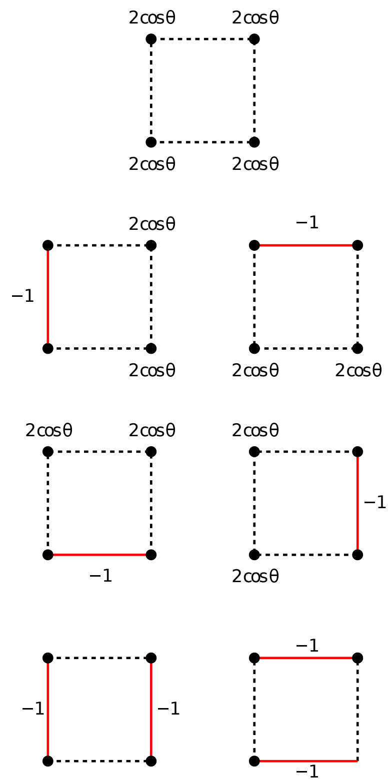

Figure 2. „Drawing" $\cos 4\theta$

and divide by 2 to obtain the formula for $\cos n\theta$. The example in Figure 2 shows how to get the formula for $\cos 4\theta$.

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

The formulas

In the previous subsections we have seen three methods to derive the expressions for $\sin n\theta$ and $\cos n\theta$, respectively. We can choose any method to obtain the following results. We list here some of the formulas for small n.

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$
$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

The first few formulas for $\sin n\theta$ are the following:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = \sin\theta(4\cos^2\theta - 1)$$
$$\sin 4\theta = \sin\theta(8\cos^3\theta - 4\cos\theta)$$
$$\sin 5\theta = \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)$$
$$\sin 6\theta = \sin\theta(32\cos^5\theta - 32\cos^3\theta + 6\cos\theta).$$

In the next sections we focus on these polynomials, seeing how they arise in different areas. The presentations of these connections should on one hand explain why the ,,drawing rule" works, on the other hand wake up the interest for these studies, see them in a wider context. In the last section we also show an example how the ,,drawing rule", the combinatorial approach can be used for advanced studies of the polynomials.

The Chebyshev polynomials

Chebyshev polynomials of first and second kind are well-studied sequences of orthogonal polynomials with many interesting properties and applications, for instance in approximation theory. In this section we collected some well-known basic facts about these polynomials that arise in several forms and ways in different areas of mathematics.

Historical remarks

Pafnuty Chebyshev (1821–1894) was a Russian mathematician, professor and a revered teacher in St. Petersburg. He has great contributions in the fields of probability theory, statistics, number theory, mechanics. Chebychev had a passion for building models and machines of new kind. The Foot-Stepping Machine was presented in the Paris World's fair in 1878. ("Chebyshevs Foot-Stepping Machine (MATH-Film 2008)", 2008) One of his main interests was the theory of mechanisms called linkages. Linkages were used to convert one type of motions into another. His goal was to find a mathematical method for systematically devising linkages to produce desired types of motions with high accuracy. This problem led Chebyshev to introduce and investigate the polynomials that were named after him. This story is a beautiful example for the gift of Chebyshev, who created from a seemingly simply setting a mathematical theory that reaches far beyond the initial application.(Albert, 2009) Chebyshev considered actually the following question in a paper of 1854. Let f be a function on $[-1, 1]$. Approximate f by a polynomial with leading coefficient 1:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1} + x^n.$$

Chebyshev defined a sequence of polynomials such that the norm

$$||p|| = \sup_{-1 \leq x \leq 1} |p(x)|$$

is minimized. mial of degree $n-1$ Another interesting fact about the Chebyshev polynomial of the first kind, T_n was conjectured first by Erdős (Erdős, 1939) in 1939. He conjectured that among all the polynomials of degree n with real coefficients and leading coefficient 1, satisfying $||p|| \leq 1$, T_n has the longest arc length on the interval $-1 \leq x \leq 1$. Quite hard proofs were given after 40 years independently by Kristiansen (Kristiansen, 1979) and Bojanov (Bojanov, 1982).

An important aspect of Chebyshev polynomials is that they are orthogonal polynomials with respect to a certain inner product defined on a vector space of functions from $[-1, 1]$ to R. The study of orthogonal polynomials is today still an active research field, and Chebyshev was the first who made investigations in full generality on this topic.

Definition and properties of Chebyshev polynomials

For a non-negative integer n, the Chebyshev polynomials of the first kind, $T_n(x)$, respectively second kind, $U_n(x)$, of degree n is defined as follows. Given any $x \in [-1, 1]$, there is a unique angle $0 \leq \theta \leq \pi$ such that $x = \cos\theta$. Then,

$$T_n(x) = \cos n\theta,$$
$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

There is a simple connection between these two polynomials:

$$T_n'(x) = nU_{n-1}(x). \tag{8}$$

A crucial property of the Chebyshev polynomials are that they can be generated recursively. (This property implies for instance the orthogonality.) Let

$T_0(x) = 1$ and $T_1(x) = x$. Then,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x); \tag{9}$$

The first few polynomials are:

$$\begin{aligned}
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 16x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}$$

The recurrence relation for the Chebyshev polynomials of the second kind is as follows:

$$U_0(x) = 1; \quad U_1(x) = 2x; \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

The first few polynomials are:

$$\begin{aligned}
U_2(x) &= 4x^2 - 1\\
U_3(x) &= 8x^3 - 4x\\
U_4(x) &= 16x^4 - 12x^2 + 1\\
U_5(x) &= 32x^5 - 32x^3 + 6x\\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1.
\end{aligned}$$

Next, we describe how to use the recurrence (9) to calculate the coefficients of the Chebyshev polynomial of the first kind in a Pascal-like triangle. Start with 1. We obtain the entries of the row from the previous row as follows. Decrease the double of the entry in the previous row directly above the entry you wish to calculate by the number left to it. More precisely, let $t_{n,k}$ denote the entry in the nth row and kth column, then to obtain $t_{n,k}$ multiply the entry $t_{n-1,k}$ and reduce it by $t_{n-1,k-1}$. The coefficients can be find in the ascendent diagonal. The following table (Table 1) shows the first values of the triangles. As an example we underlined the coefficients of $T_5(x)$ and $U_5(x)$.

Similar method can be used for the coefficients of the Chebyshev polynomials of the second kind. We mention here that Macdougall (Macdougall, 1999) presented a Pascal-like triangle for the generation of the absolute values of the coefficients of both Chebyshev polynomials simultaneously based on the adding

1								1						
1	-1							2	-1					
2	-3	1						4	-4	1				
4	-8	5	-1					8	-12	6	-1			
8	-20	18	-7	1				16	-32	24	-8	1		
16	-48	56	-32	9	-1			32	-80	80	-40	10	-1	
32	-112	160	-120	60	-11	1		64	-192	240	-160	60	-12	1

Table 1. Calculations of the coefficients

formula of trigonometric functions combined with de Moivre formula. It is easier to read the polynomials from this triangle off, if we know some basic facts about them. The Chebyshev polynomial of the first kind, $T_n(x)$, is a polynomial of degree n, with leading coefficient 2^{n-1}. The Chebyshev polynomial of the second kind, $U_n(x)$, is a polynomial of degree n with leading coefficient 2^n. It is true for both sequences of polynomials, $T_n(x)$ and $U_n(x)$, that they have only even powers of x if n is even, and only odd powers of x, if n is odd. The roots of the Chebyshev polynomials of the first kind are known as Chebyshev points in approximation theory. $T_n(x) = 0$ if and only if $\cos n\theta = 0$. This is true, if

$$\cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n.$$

Hence, we have

$$T_n(x) = 2^{n-1} \prod_{k=1}^{n} \left(x - \cos\left(\frac{2k-1}{2n}\pi\right)\right), \quad \text{for } n \geq 1.$$

Similarly, we have a closed product form for the Chebyshev polynomials of the second kind:

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k}{n+1}\pi\right)\right), \quad \text{for } n \geq 1.$$

There are several identities involving Chebyshev polynomials. Here we mention as an example the composition rule, which is a direct consequence of its definition by the cosine function.

$$T_n(T_m(x)) = T_{nm}(x).$$

Clearly, the trigonometric identities and identities of Chebyshev polynomials go hand in hand. We recall two examples:

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta\cos n\theta \quad \Longleftrightarrow \quad U_n(x) - U_{n-2}(x) = 2T_n(x)$$

$$\sin(n+1)\theta - \cos\theta \sin n\theta = \sin\theta \cos n\theta \quad \Longleftrightarrow \quad U_n(x) - xU_{n-1}(x) = T_n(x).$$

Matching polynomials

We have seen in the drawing rule, that matchings of paths and cycles help us to get the expressions for $\cos n\theta$ and $\frac{\sin(n+1)\theta}{\sin\theta}$. In order to understand why this works, one should be familiar with some fundamental ideas of graph theory and enumerative combinatorics. Here, we give brief description about some simple ingredients from these great disciplines (Shi, Dehmer, Li, & Gutman, 2017).

Historical remarks

After the introduction of the first graph polynomials by J. J. Sylvester in 1878, the theory of graph polynomials have been developed and have been proven useful in discrete mathematics, and in several applications as in engineering, information sciences, mathematical chemistry, etc. The matching polynomial is one of the graph polynomials that has deep-lying algebraic properties. The concept of matching polynomials in graph theory were used first by E. Farrell in 1979 (Farrell, 1979), though these polynomials arose already before in theoretical physics as a mathematical model for phase transition based on dimers (diatomic molecules) on a lattice (Gutman, Milun, & Trinajstić, 1975). All the main results were developed between 1970s and 1980s. One of the main results is that all zeros of the polynomials are real-valued. This was first proven by C. J. Heilmann and E. H. Lieb (Heilmann & Lieb, 1970). It has a very important consequence in the theoretical physic dimer model mentioned above. Since for a phase transition a complex valued zero is a condition, it followed that phase transition is not possible in this model, hence, this model was abandoned.

Notations and definition

Let G be a simple graph, (no multiple edges, loops, or directed edges) with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$, so it has $n = n(G) = |V(G)|$ vertices and $m = m(G) = |E(G)|$ edges. n is called the *order* of the graph. A *matching* of a graph G is a set of pairwise disjoint edges of G, and a $k-$*matching* is a set of k independent edges, edges that do not share endvertices. Let $m(G,k)$ denote the number of k-matchings. Clearly, $m(G,1) = m$ and $m(G,k) = 0$ for all $k > \frac{n}{2}$. It is convenient to set $m(G,0) = 1$. The *matching polynomial* of a graph G of order n is defined as

$$M(G) = M(G,x) := \sum_{k\geq 0} (-1)^k m(G,k) x^{n-2k}.$$

A more suggestive way (emphasizing its combinatorial nature and using the usual notation of weights in enumerative combinatorics) to give this definition is the following:

$$M(G,x) := \sum_{M} (-1)^{|M|} x^{ip(M)},$$

where the sum is taken over all matchings of the graph G. Here $ip(M)$ denotes the number of isolated points of M and $|M|$ the number of edges in the matching. Clearly, the two definitions differ only in their forms.

Since $m(G,k) = 0$ for $k > \frac{n}{2}$, $M(G,x)$ is a polynomial in the variable x and since $m(G,0) \neq 0$, $M(G,x)$ is a polynomial of degree n. Moreover, the polynomial is monic.

Basic properties

The matching polynomial is a graph invariant, i.e., if G and G' are isomorphic, then $M(G) = M(G')$. Moreover, if we denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ the zeros of the matching polynomial, which are all real numbers, it holds

$$\sum_{j=1}^{n} \mu_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \mu_j^2 = 2m.$$

It is not an accident that the same formulas hold for the eigenvalues of graphs, which are defined as the eigenvalues of the adjacency matrix of graphs. Namely, the theory of matching polynomials can be embedded into spectral graph theory (Shi et al., 2017).

Here, we want to emphasize the recursive calculation of matching polynomials. It is clear, that if G consists of two (disconnected) components H_1 and H_2; $G \cong H_1 \cup H_2$, then

$$M(G) = M(H_1)M(H_2),$$

or generally for a graph G consisting of r components $G \cong H_1 \cup \cdots \cup H_r$:

$$M(G) = \prod_{j=1}^{r} M(H_j).$$

So, for instance if the graph is the set of n isolated points, i.e., $|E(G)| = 0$, then $M(G) = x^n$. For the next results we need some more notations. If u is a vertex of G we let $G - u$ denote the graph obtained by deleting the vertex u and all its incident edges. If e is an edge of G, then $G - e$ is the subgraph of G obtained by deleting the edge e but keeping all the vertices. If $e = \{u, v\}$ then we write $G - [e]$ for $G - u - v$. The following relation is an easy but fundamental observation.

Proposition 2. *For the number of k-matchings of a graph G, the following recursion holds:*

$$m(G, k) = m(G - e, k) + m(G - [e], k - 1). \tag{10}$$

Proof. A set of k independent edges contains e or does not contain e. The number of sets of k independent edges not containing e is clearly $m(G - e, k)$, while the number of those containing the edge e is the same as the number of independent sets of $k - 1$ edges selecting from the graph obtained by deleting the edge e, which number we denoted by $m(G - [e], k - 1)$. □

A direct consequence of (10) is the following:

Proposition 3. *Given a graph G, the matching polynomial satisfies:*

$$M(G) = M(G - e) - M(G - [e]). \tag{11}$$

Proof.

$$\begin{aligned}
M(G) &= \sum_{k\geq 0}(-1)^k m(G, k)x^{n-2k} \\
&= \sum_{k\geq 0}(-1)^k [m(G - e) + m(G - [e], k - 1)]x^{n-2k} \\
&= \sum_{k\geq 0}(-1)^k m(G - e)x^{n-2k} \\
&+ (-1)\sum_{k\geq 0}(-1)^{k-1} m(G - [e], k - 1)x^{(n-2)-2(k-1)} \\
&= M(G - e) - M(G - [e]).
\end{aligned}$$

Note, that $|V(G - [e])| = n - 2$. □

In particular, if $v \in V(G)$ with degree 1, attached to a vertex u, (in particular if $e = \{u, v\}$ and v has no other neighbours), we have

$$M(G) = xM(G - v) - M(G - u - v). \tag{12}$$

Generally, if $v \in V(G)$ and the neighbours of v are $\{u_1, \ldots, u_d\}$, then

$$M(G) = xM(G - v) - \sum_{i=1}^{d} M(G - u_i - v).$$

We mention two more interesting identities:

$$\sum_{v \in V(G)} M(G - v) = M'(G) \quad \text{and} \quad M(G) = \sum_{v \in V(G)} \int_0^x M(G - v)dx + M(G, 0),$$

where $M(G, 0)$ is the value of $M(G)$ at $x = 0$.

Matching polynomials of paths and cycles

We focus now on the matching polynomial of two simple graphs, the path graph and the cycle graph. We let P_n denote the *path graph* with with $V(G) = \{v_1, \ldots v_n\}$ and $E(G) = \{e | e = \{v_i, v_{i+1}\}, i = 1, \ldots, n-1\}$. An *n-cycle* is denoted by C_n, and defined by $V(G) = \{v_1, \ldots v_n\}$ and $E(G) = \{e | e = \{v_i, v_{i+1}\}, i = 1, \ldots, n-1\} \cup \{e = \{v_1, v_n\}\}$. According to (12) the matching polynomials of the paths P_n, $n \geq 0$ satisfy the recurrence relation

$$M(P_n) = xM(P_{n-1}) - M(P_{n-2}),$$

with the initial conditions

$$M(P_0) = 1, \quad \text{and} \quad M(P_1) = x.$$

The first few polynomials are:

$$\begin{aligned}
M(P_2) &= x^2 - 1 \\
M(P_3) &= x^3 - 2x \\
M(P_4) &= x^4 - 3x^2 + 1 \\
M(P_5) &= x^5 - 4x^3 + 3x \\
M(P_6) &= x^6 - 5x^4 + 6x^2 - 1 \\
M(P_7) &= x^7 - 6x^5 + 10x^3 - 4x
\end{aligned}$$

The case of n-cycle is immediate if we notice that for any edge e of C_n we have $C_n - e = P_n$ and $C_n - [e] = P_{n-2}$. After applying the recurrences (11) and (12), we obtain the relation:

$$M(C_n) = M(P_n) - M(P_{n-2})$$

$$
\begin{aligned}
&= [xM(P_{n-1}) - M(P_{n-2})] - [xM(P_{n-3}) - M(P_{n-4})] \\
&= x\,[M(P_{n-1}) - M(P_{n-3})] - [M(P_{n-2}) - M(P_{n-4})] \\
&= xM(C_{n-1}) - M(C_{n-2}).
\end{aligned}
$$

The initial values are $M(C_1) = x$, $M(C_2) = x^2 - 2$, and some further polynomials are

$$
\begin{aligned}
M(C_3) &= x^3 - 3x \\
M(C_4) &= x^4 - 4x^2 + 2 \\
M(C_5) &= x^5 - 5x^3 + 5x \\
M(C_6) &= x^6 - 6x^4 + 9x^2 - 2 \\
M(C_7) &= x^7 - 7x^5 + 14x^3 - 7x.
\end{aligned}
$$

The recursions

The similarity between Chebyshev polynomials and matching polynomials of paths and cycles is not an accident. The relation between them is precisely the following:

$$
\begin{aligned}
T_n(x) &= \frac{1}{2}M(C_n, 2x), \\
U_{n+1}(x) &= M(P_n, 2x).
\end{aligned}
$$

We have seen that in both cases (Chebyshev polynomials and matching polynomials) the recursive relation plays a key role and the above mentioned connection can be proven for instance by the recursions. Sequences of numbers and polynomials that fulfill such three term recurrences are studied in combinatorics, algebra, geometry and combinatorics for their own sake. For instance, Fibonacci polynomials are defined by the recursion

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

for $n \geq 0$ with $F_0(x) =$, $F_1(x) = 1$. The first few polynomials are:

$$
\begin{aligned}
F_2(x) &= x^2 + 1 \\
F_3(x) &= x^3 + 2x \\
F_4(x) &= x^4 + 3x^2 + 1
\end{aligned}
$$

Setting $x = 1$, the formula reduces to the well-known Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 34, 55, 89, 144, \ldots.$$

The same recursion with different starting values defines the Lucas polynomials and sequence, respectively.

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$$

for $n \geq 0$ with $L_0(x) = 2$, $L_1(x) = x$. The first few polynomials are:

$$L_2(x) = x^2 + 2$$
$$L_3(x) = x^3 + 3x$$
$$L_4(x) = x^4 + 4x^2 + 2$$

$L_n(1)$ gives the Lucas numbers:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots.$$

Again, here we see also the analogue. $M(P_n)$ and $M(C_n)$ obey the same recurrence with different initial values. It is well known that the Fibonacci number F_{n+1} counts for instance the 1's and 2's which sums to n. Equivalently, (by a straightforward bijection) the number of matchings of a path of length n, while the Lucas number L_n counts the matchings of a cycle of length n. The connection to our problem is quite obvious. Fibonacci numbers are one of the most studied number sequences: around 100 identities involving Fibonacci numbers are known. This wealth motivated several mathematicians to consider generalizations of Fibonacci numbers. One line of such researches is the study of polynomials that are generally defined by the recursion relation.

$$< n + 2 >= x < n + 1 > +y < n >,$$

for $n \geq 0$. For instance, if $< 0 >= 0$ and $< 1 >= 1$ the first values are:

$$< 2 >= x, \quad < 3 >= x^2 + y, \quad < 4 >= x^3 + 2xy, \quad < 5 >= x^4 + 3x^2y + y^2$$

Varying the initial values, and special settings of x and y, the sequence reduces to different important sequences (involving Chebyshev polynomials). We mention here just some examples. For $< 0 >= 0$, $< 1 >= 1$, $x = 2$, $y = -1$ it reduces to the non-negative integers $< n >= n$, for $< 0 >= 0$, $< 1 >= 1$, $x = 1 + q$,

$y = -q$ to the standard q-integers $< n >= [n]_q = \frac{1-q^n}{1-q}$. For $< 0 >= 0$, $< 1 >= 1$, $x = 2$, $y = 1$ the sequence reduces to the Pell numbers, that are known as the denominators of the closest rational approximations to the square root of two: $0, 1, 2, 5, 12, 29, 70, 169, \ldots$. Setting $< 0 >= 1$, $< 1 >= 2x$, $x- > 2x$ $y = -1$ we obtain the Chebyshev polynomials of the second kind, while setting $< 0 >= 1$, $< 1 >= x$, $x- > 2x$, $y = -1$ to the Chebyshev polynomials of the first kind.

We see that these sequences and polynomials are in strong connections, fall under the same family.

We mention in this last section that one can use this combinatorial approach for deriving and proving trigonometric identities as we see in a series of paper of Benjamin and Quinn (Benjamin & Walton, 2009; Benjamin & Quinn, 2003; Benjamin, Ericksen, Jayawant, & Shattuck, 2010). As a simple example we prove the closed form of $U_n(x)$:

Proposition 4.

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}$$

Proof. The combinatorial interpretation tells us that this formula counts the matchings of a path of length n such that every edge is associated with the weight -1 and every isolated point with $2x$. On the right side we see a sum, where clearly k stands for the number of edges in the considered matching. So on the right side we count the matchings according to the number of edges, and then we sum over all possibilities from $k = 0$ to $k = \lfloor \frac{n}{2} \rfloor$. If a matching has k edges, then $n - 2k$ isolated vertices remain. (The weights show this situation $(-1)^k$ and $(2x)^{n-2k}$). The question is how many matchings with k edges are there? Consider the first $n - k$ vertices of the path and choose k vertices out of these. Insert (shift one of the vertices $n - k + 1, n - k + 2, \ldots, n$ one after the other) one vertex after each chosen vertex and mark the edge between the chosen and the inserted vertex for the matching. □

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