17/1 (2019), 59–72
DOI: 10.5485/TMCS.2019.0457
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Teaching
Mathematics and
Computer Science

Square root in secondary school

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Abstract. Although in Hungary, for decades, the calculation method of the square root of a real number is not in the mathematics curriculum, many of the taught concepts and procedures can be carried out using different square root finding methods. These provide an opportunity for students in secondary school to practice and deepen understand the compulsory curriculum. This article presents seven square-root-finding methods, currently teachable in secondary schools.

Key words and phrases: square root, history of mathematics, secondary school curriculum.

ZDM Subject Classification: A33, A34, F53, F54.

Introduction

In mathematics education we first learn about numbers, second we get acquainted with operations on them. After working out simple mental calculations, we learn written methods used for complex operations. The first operations where the result is not calculated by ourselves (we look it up in tables, or calculate it by a calculator) is the square root. Later the demand for individual calculation is lost, no student asks how to find the logarithm of a number with pen and paper. Since the topic of square root is close to the four basic operations which we can do using formal written methods, it comes into a lot of students' mind if there is a pen-and-paper method to calculate the square root of a number. As a high school teacher I often wish that the students were as interested in compulsory topics as in these ones for fun.

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In this article we are going to show seven square root finding methods that are suitable for secondary school students and by which we, beside working out square roots, can improve skills and competences that are compulsory parts of the national curriculum providing they are discussed at the appropriate time and in the appropriate depth in classes or workgroups. There are several consequences of this aim. First those square root finding methods that are beyond the level of final exam are not discussed. Second, in each case those parts of the methods are presented that can help to acquire the compulsory curriculum. Sometimes it is the mathematical background of the given method or it is just a simple algorithm. However, we always mention where the detailed proof can be found.

Why is it important for a teacher to know these kinds of methods? According to Gábor Tákacs: *"The acquisition of mathematical skills and competences requires solution of several calculation exercises. This can become monotonous since repeating an activity loses positive influence and motivation, rooted on the activity itself, fades away. Obviously, ideally the subject of the activity, the mathematical problem, has the function of motivation at the same time."* (Takács, 2006, p. 3)

The following square-root finding methods satisfy these needs. So after presenting these methods a list of skills and competences are given that can be developed by finding square-root. On one hand it satisfies the students' natural curiosity, on the other hand the topic that should be taught can be practised in different situations which can help its deeper understanding and it's saving into long-term memory.

Each technique is followed by an example. The numbers used here were chosen to emphasize the properties of the given method. For example, in some cases we compute the square root of five or six-digit numbers, in other cases we have chosen a two digit number to make the calculation easier. In addition to this, if we had calculated the square root of the same number in each case, the reader would believe automatically that one method is better than the other one since we get more exact result than in the other case while it can only be thanked to a luckier choose.

These methods were tried out in Hungary, where the school system is divided into two levels: eight-year primary school followed by a 3-year vocational school, a 4-year vocational secondary school where vocational subjects and basic subjects are taught, or a 4-year (5-year in special bilingual classes) secondary school that prepares students for higher education. Mathematics is taught in each grade. At the compulsory school final exam students should choose between two levels in

Mathematics: standard level and higher level, which requires higher knowledge and good problem-solving skills. Although it is possible to gain admission to a lot of university faculties with both types of exams, in some cases the higher level exam is compulsory. The practice of workgroups, where the development of talented students takes place should be also mentioned. These are not compulsory classes, so they are not restricted by the national curriculum. Students there can hear about higher level topics or practise complex problems once or twice a week. The following techniques can help students' and teachers' work at compulsory education and workgroups as well.

János Szendrei classifies the results of mathematics research into three categories. Among them, *"The third category could be the category of enhancers of practical efficiency. Teachers are delighted with the research that helps them understand what they teach and give them ideas for teaching."* (Szendrei, 1993, p. 17) We believe our work is in this category.

Finding square root by hand

The written method of finding the square root (like the other operations) was part of the curriculum a long time ago, it was taught as an algorithm. However, it is not advisable to show the proof to the students. We can find a demonstration idea for students in "idea why the algorithm is good[1]".

The algorithm consists of the following steps:

Step 1: Group the numbers under the root in pairs on both sides of the decimal point (the left-side group may be one-digit, on the right side 0 should be written if needed).

Step 2: Find the integer part of the leftmost group's square root, and write it down after the equal sign. This will be the first number of the result.

Step 3: Subtract the square of this number from the first group, move the second group down next to the difference.

Step 4: Double the already found root, write an unknown number to it, to get the biggest number that is not bigger than the second number. The unknown number is the next digit of the square root.

Step 5: Then repeat the third and the forth steps until you like. As difficult this method seems in theory as simple it is in practise.

[1]Retrieved from `http://villemin.gerard.free.fr/ThNbDemo/RacinCar.htm#marche`

Problem: Find $\sqrt{312}$ to one decimal place.

$\sqrt{3|12,00|00|...} \approx \overline{ab,cd}$

$1^2 \leq 3$ et $2^2 > 3$. So $a = 1$

$3 - 1^2 = 2$

$\overline{2b} \cdot b \leq 212$. So $b = 7$ since $27 \cdot 7 = 189$ (and $28 \cdot 8 = 224 > 212$)

$212 - 189 = 23$

$\overline{34c} \cdot c \leq 2300$. So $c = 6$ since $346 \cdot 6 = 2076$ (and $347 \cdot 7 = 2429 > 2300$)

$2300 - 2076 = 224$

$\overline{352c} \cdot c \leq 22400$. So $c = 6$ since $3526 \cdot 6 = 21156$ (and $3527 \cdot 7 = 24689 > 22400$)

Then $\sqrt{312} \approx 17,66 \approx 17,7$.

Since this method requires only the four basic operations, it can be shown even to 8th grade students as well. The main profit of using this method is that two of the basic operations are worked out in writing and estimation which based on my experiences is difficult for students, can be practiced.

Other advantages of this method is that the found decimal (unlike other approximation methods) should not be corrected. With some practise it can be easily learnt, and the calculation (till 3-4 digits) can be worked out quickly. The method should be presented in secondary school as well if students are preparing for a competition where calculators are not allowed to use. The algorithmic structure of the method suits well to the concept of basic operations. Proof of the algorithm is not worth discussing (unlike the long division). It is better if we regard this method just a tool.

Finding square root with odd numbers' sum (in French: Méthode Du Goutte Á Goutte, The Dropping Method)

In Hungary this method is not widely known. It is based on the fact, that the sum of consecutive odd numbers starting from 1 creates square numbers. This fact is taught in secondary schools. Like in previous case it is not worth discussing the proof of the algorithm described in "Root extraction technique[2]".

The steps of the method:

Step 1: The given number is grouped into pairs on both sides of the decimal point (the leftmost group may be one-digit).

[2]Retrieved from `http://fracademic.com/dic.nsf/frwiki/1608983#Preuve_de_l.27algorithme`

Step 2: From the leftmost group we subtract the odd numbers starting from one (1, 3, 5 ...) until we get a non-negative number.

Step 3: The result of subtraction gives the first digit of square root.

Step 4: Take the last subtracted odd number, add one, multiply it by 10 and add 1.

Step 5: Put the next number group to the difference we have got in step 2.

Step 6: From this number subtract the consecutive odd numbers starting from the number got in step 4 until we get a non-negative number.

Step 7: The result of subtractions gives the next digit of square root.

Continue this process as long as you like (put 00 to the next number group if needed) to find the next digit. If in step 3 after subtraction the difference is 0 and there are only 0 digits in the number we stop and we put as many zeros to the square as the number of the remaining number groups. If we can compute 0 subtraction, we deduct the number got in step 4 by 2, and we regard it as the last used odd number.

Problem: Find $\sqrt{715}$ expressed to one decimal place.

Step 1: We split the number: $7|15$

Step 2: $1+3=4$. We approximate 7 with this (since $1+3+5>7$). Since this is a two-term sum, $\sqrt{715} \approx \overline{2a}$.

Step 3: $7-4=3$, so we approximate 315.

Step 4: Since $(3+1)\cdot 10+1=41$, 315 is approximated by this sum $41+43+45+47+49+51=276$. Since it is a six-term sum, the next number of the root is 6, so $\sqrt{715} \approx 26$.

Step 5: (Repetition of Step 3.): $315-276=39$, so we approximate 3900.

Step 6: (Repetition of Step 4.): Since $(51+1)\cdot 10+1=521$, the approximation of 3900 is $521+523+525+527+529+531+533=3689$. Since it is a seven-term sum, $\sqrt{715} \approx 26,7$.

Step 7: (Repetition of step 3.): $3900-3689=211$, so we try to approximate 21100.

Step 8: (Repetition of step 4.): Since $(533+1)\cdot 10+1=5341$, so 21100 is approximated by $5341+5343+5345=16029$ sum. Since it is a three-term sum, $\sqrt{715} \approx 26,73$.

Step 9: Then expressed to one decimal place $\sqrt{715} \approx 26,7$.

The method is interesting, but it is not really suitable for daily use. However, it really helps the development of algorithmic thinking, acquisition of longer algorithms, practice of basic operations, or (with small modifications) revision of the sum of consecutive terms of an arithmetic sequence. Due to this and to the complexity of the method it is suitable rather for 12th grade students.

Finding square root with continued fractions

Rafael Bombelli (1526-1572) Italian mathematician in his book called L'Algebra published in 1579 (Figure 1[3]) finds the square root of 13 with infinite continued fractions ($*$).


L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copiosa delle materie, che in essa si contengono.
Posta hora in luce à beneficio delli Studiosi di detta professione.
IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori

Figure 1. The title page of Bombelli's work

$$\sqrt{13} = 3 + x \quad (0 < x < 1) \quad (*)$$

Squaring both sides: $13 = 9 + 6x + x^2$, from where $4 = x(x+6)$

$$\frac{4}{6+x} = x$$

$$13 = 3 + \frac{4}{6 + \frac{4}{6+\frac{4}{6+\dots}}}$$

[3]source: `http://www-history.mcs.st-and.ac.uk/`

If we calculate the value of each fraction we get the following sequence converging to $\sqrt{13}$ 3; $3+\frac{4}{6}=3,6$; $3+\frac{4}{6+\frac{4}{6}}=3,6$; $3+\frac{4}{6+\frac{4}{6+\frac{4}{6}}}=3,60$; $3+\frac{4}{6+\frac{4}{6+\frac{4}{6+\frac{4}{6}}}}\approx 3,6055045872$.

For comparison (using calculator) $\sqrt{13}\approx 3,605551275$, so the fifth term of our sequence creates the value of $\sqrt{13}$ correct to four decimal place. We note that for a less lucky number the convergence is not so fast.

It is important to point out that Bombelli's continued fraction defines a sequence that converges to $\sqrt{13}$. However, it does not mean that it is the only sequence with this property. It is also important to make the students aware of the fact that finding a thing with a given property does not mean that we have found all of them. (Remember the solution of an equation. Looking at the equation and finding a root do not mean that we have found all the roots.) With Euler's (1707-1783) method we can easily create another sequence converging to $\sqrt{13}$. If $x=\sqrt{13}$, then $x^2=13$ adding x to both sides :

$$x^2+x=13+x$$
$$x(x+1)=(x+1)+12$$
$$x=1+\frac{12}{1+x}$$
$$\sqrt{13}=1+\frac{12}{2+\frac{12}{2+\frac{12}{\cdots}}}$$

The sequence we get: 1; 7; $\frac{5}{2}=2,5$; $\frac{31}{7}\approx 4,43$; $\frac{61}{19}\approx 3,21$; $\frac{77}{10}\approx 3,85$.

The general description of the method with its proof which is suitable for secondary school workgroups is in work of Gergely (1973) and Bombelli.

According to my experiences it is useful to show this method to students in 9th grade when we teach algebraic fraction. It breaks the monotony of the lesson, we can answer students' frequently asked question (What is it good for?) by this so it might give some meaning to the existence of the continued fractions for them. While using Bombelli's method we come across different topics from the curriculum like estimation (in certain classes the strictly monotonous root functions can be mentioned), restrictions on variables, square of a two-term sum, solving equations, factorising, then the calculation of the continued fractions. During calculation we can put down the base of the terms like error term, convergent sequence, rates of convergence, oscillating sequences.

In workgroups we can mention that with the help of this method we can estimate the value of irrational numbers through sequences of rational numbers. In classes specialised in mathematics where we, in Hungary, are allowed to choose an optional topic, it is suitable for an introduction of abstract algebra or measure theory to point out that the rational number set is not complete (there is non-convergent Cauchy-sequence on $\mathbb{Q}$).

Finding square root with Gauss Mean

The arithmetic and geometric mean as well as the ration between them are an integral part of the secondary school curriculum. Gaussian means (e.g. arithmetic-geometric, geometric-harmonic, etc.) are less known while harmonic-arithmetic is suitable for defining a sequence converging (fairly quickly) to $\sqrt{a}$. Let $0 < a < b$ be given real numbers, x_n, y_n are sequences to which

$$x_1 = a, \quad y_1 = b, \quad x_{n+1} = \frac{2x_n y_n}{x_n + y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2} \quad (n \in \mathbb{N}^+)$$

In Lajkó's and Urbán's work (Lajkó, 2006; Urbán, 2004) there is the proof of that $\lim\limits_{n\to\infty} \{x_n\} = \lim\limits_{n\to\infty} \{y_n\} = \sqrt{ab}$. The common limit of these sequences is called the harmonic-arithmetic mean of a and b numbers. The proof is not over the expectations of the higher level final exam, so it can be presented on elective courses or on workgroups for talented students.

Problem: Define a a_n sequence converging to $\sqrt{40}$. Give the first seven terms of the sequence.

$$a_1 = x_1 = 5$$

$$a_2 = y_2 = 8$$

$$a_3 = x_2 = \frac{2 \cdot 5 \cdot 8}{5+8} = \frac{80}{13} \approx 6,1538462$$

$$a_4 = y_2 = \frac{5+8}{2} = \frac{13}{2} = 6,5$$

$$a_5 = x_3 = \frac{2 \cdot \frac{80}{13} \cdot \frac{13}{2}}{\frac{80}{13} + \frac{13}{2}} = \frac{2080}{329} \approx 6,322188$$

$$a_6 = y_3 = \frac{\frac{80}{13} + \frac{13}{2}}{2} = \frac{329}{52} \approx 6,3269231$$

$$a_7 = x_4 = \frac{2 \cdot \frac{2080}{329} \cdot \frac{329}{52}}{\frac{2080}{329} + \frac{329}{52}} = \frac{1368640}{216401} \approx 6,324555$$

As a comparison (using calculator) $\sqrt{40} \approx 6,32455532$. So the seventh term of our sequence gives the value of $\sqrt{40}$ expressed to six decimal place.

Estimation using linear function approximation

This method is based on topics from 9th grade and can be discussed with special products (factorising the difference of two squares), or with linear functions or even with linear transformation as found in Pálfalvi's work (Pálfalvi, 1990) expressly written for the 12-14 years old children. It is clear from the above examples that this method gives great opportunity to integrate the different fields of mathematics in students' mind.

The other advantage of this method is that it is also suitable for introduce estimation and error calculation in workgroups, which can make the base of topics in future education like the term linear interpolation.

Let's start with the well-known special product: $a^2 - b^2 = (a-b)(a+b)$.

Then $a = b + \frac{a^2-b^2}{a+b} (*)$, if $a \neq -b$.

So, if we are looking for $\sqrt{n^2+d}$ where $d \in \mathbb{R}^+$, and $n < \sqrt{n^2+d} < n+1$, then from $(*)$ with substitution $a = \sqrt{n^2+d}, b = n$, we get

$$\sqrt{n^2+d} = n + \frac{n^2+d-n^2}{\sqrt{n^2+d}+n} = n + \frac{d}{\sqrt{n^2+d}+n} > n + \frac{d}{2n+1}$$

So, if the value of $\sqrt{n^2+d}$ is approximated by $n + \frac{d}{2n+1}$ then the error is:

$$\sqrt{n^2+d} - (n + \frac{d}{2n+1}) = \frac{d}{\sqrt{n^2+d}+n} - \frac{d}{2n+1} < \frac{d}{2n} - \frac{d}{2n+1} = \frac{d}{2n(2n+1)}$$

Problem: Estimate the value of $\sqrt{75}$ with four decimal places.

$$\sqrt{75} = \sqrt{64+11} \approx 8 + \frac{11}{2 \cdot 8 + 1} \approx 8,6471$$

The error is less than $\frac{11}{2 \cdot 8 \cdot 17} \approx 0,04$.

Plotting the square root function on the appropriate interval (Figure 2) the estimation and the error themselves can be presented nicely.

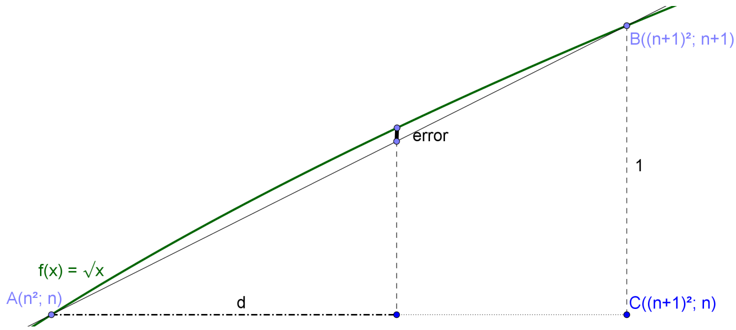

Figure 2

From the similar triangles on the Figure 2 we get the above result since $\frac{y}{d} = \frac{1}{2n+1}$, then $y = \frac{d}{2n+1}$, so $\sqrt{n^2+d} \approx n + \frac{d}{2n+1}$.

It is important to point out that unlike previous ones it is not a classical square root finding method where repeating the steps we get more and more exact values, it is an estimation of the value of the square root.

Finding square root through solving equation with Newton-method

Finding the value of $\sqrt{a}$ $(a \in \mathbb{R}^+)$ can be regarded as finding the positive root of $f(x) = x^2 - a$ function, that is why we can use the so called Newton (or Newton-Raphson or Newton-Fourier) root finding method that was worked out by Isaac Newton and which later was generalised and improved by Wallis, Joseph Raphson and Thomas Simson.

If the method is used to $f(x) = x^2 - a$ function, first we find x_0, where $0 \leq x_0 - 1 < \sqrt{a} < x_0$ and $x_0 \in \mathbb{Z}$. We determine the tangent line of $f(x)$ through $x = x_0$ call it e_1. Find the intersection of e1 with x axis (this is the solution of a linear equation) denote by x_1. We determine the tangent line of $f(x)$ that goes through $x = x_1$ call it e_2. Find the intersection of e_2 with axis x call it x_2, and so on (Figure 3). The $x_1, x_2, x_3, \ldots$ sequence we have got converges to the zero of function $f(x)$ (in this case to $\sqrt{a}$).

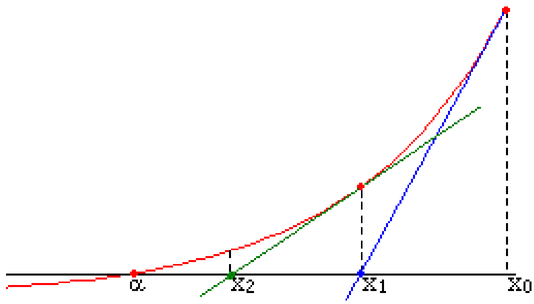

Figure 3

Problem: Estimate the value of $\sqrt{90}$ with the first two steps of the Newton method.

Find the positive zero of $f(x) = x^2 - 90$ function. We know $9 < \sqrt{90} < 10$.

The equation of the tangent line of $y = f(x)$ curve through abscissa=10 is: $y = 20x - 190$. It intercepts x axis at $x = 9,5$.

The equation of the tangent line of $y = f(x)$ curve through abscissa= 9,5 is $y = 19x - 180,25$.

The intercept of x axis is $x = \frac{180,25}{19} \approx 9,4868$.

For comparison $\sqrt{90} \approx 9,486832981$ (using calculator).

The method is worth presenting on elective courses (or workgroups for 11th-12th grade students) when they learn about equation of tangent line. The lesson becomes more interesting if students can see that creating equation of a tangent is not only the goal of an exercise, but it can be a tool for solving a problem also. In addition to this, another field of usage of differential calculus can be presented.

When we generalise the method to find the zero of other types of functions, the disadvantages of this technique appear (e.g. starting the iteration from too far, we cannot get the appropriate sequence) and we have to discuss the function's complexity (from which side should we start the iteration?) which is less known in secondary school.

Finding square root eith Newton (Babilonian or Heron) method

Most Hungarian textbooks and learning materials (e.g. all collections of figures that are allowed to use in secondary school) connects the method based on recursive sequences to Newton. However, it actually appeared in ancient times in Babilonian mathematics and was published in the first book of Metrica, Heron of Alexandria's work, found in 1896.

Take the following recursive sequence $x_0 = [\sqrt{a}], x_{k+1} = \frac{1}{2}(x_k + \frac{a}{x_k})$. As it can clearly see from the proof in Sain's work (Sain, 1986) $\{x_n\}$ converges really fast to $\sqrt{a}$ (x_{k+1} has twice as much correct place than x_k.)

Problem: Find the value of $\sqrt{20}$ expressed to four decimal place.

$$x_0 = [\sqrt{20}] = 4$$

$$x_1 = \frac{1}{2}\left(4 + \frac{20}{4}\right) = \frac{9}{2}$$

$$x_2 = \frac{1}{2}\left(\frac{9}{2} + \frac{20}{\frac{9}{2}}\right) = \frac{161}{36} \approx 4,472222$$

$$x_3 = \frac{1}{2}\left(\frac{161}{36} + \frac{20}{\frac{161}{36}}\right) = \frac{51841}{11592} \approx 4,472136$$

While (using calculator) $\sqrt{20} \approx 4,472136$.

After examining the model, we can realize that this is actually a special case of Newton's iteration (That is why we call it Newton type square root finding method.) The sequence we get $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, where if $f(x) = x^2 - a$, then $x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{x_k^2 + a}{2x_k} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$.

This method–if we choose appropriate depth and place–can be a part of a lesson or a workgroup topic.

The rate of convergence and the simplicity of the formula make this method the most frequently used square root finding technique in Hungary.

Summary

The methods presented above are all fit into Hungarian national curriculum and from my experiences I can state that they are suitable for motivating students and supporting their learning. However, teacher knowing his or her class should decide which method to choose. To help this decision the methods and the topics from the curriculum related to them are collected.

Method	Topic from the curriculum
1. Finding square root in writing	definition of square root calculation (practice of basic operations)
2. Finding square root with odd numbers' sum	the sum of the first n terms of an arithmetic sequence
3. Continued fraction method	algebraic fractions
4. Square root with Gaussian mean	special means limit of a sequence
5. Estimation with a linear function	linear function, similarity
6. Newton-iteration	equation of tangent convexity
7. Square root finding with Newton (Babilonian or Heron) method	limit of a sequence

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(Received November, 2018)