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Infimum problems derived from the proofs of some generalized Schwarz inequalities

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Abstract. We define

$$f_{(a,b)}(r) = a\,r + b/r$$

for all $a, b, r \in \mathbb{R}$ with $r > 0$.

And, for some subsets A of $\mathbb{R}$, we determine

$$F_{A_+}(a, b) = \inf_{r \in A_+} f_{(a,b)}(r),$$

where $A_+ = \{ r \in A : \ r > 0 \}$.

The above infima are mainly motivated by the proofs of some recent generalized Schwarz inequalities established by the present authors.

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1. A motivation for the determination of the infimum $F_{\mathbb{R}_+}(a, b)$

In the sequel, we shall use the standard notations $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ for the sets of all real, natural, integral and rational numbers, respectively.

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Moreover, for a fixed $k \in \mathbb{N}$, we denote by $\mathbb{R}^k$ the family of all k–tuples

$$x = (x_i)_{i=1}^k = (x_1, x_2, \ldots, x_k)$$

such that $x_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, k$. Thus, x is actually a function of the set $\{1, 2, \ldots, k\}$ to $\mathbb{R}$.

As it is usual, for any $x, y \in \mathbb{R}^k$, we define

$$x + y = \big(x_i + y_i\big)_{i=1}^k .$$

Thus, it can be easily seen that $\mathbb{R}^k$ is an abelian group in the sense that

(1) $x + y = y + x$; (2) $x + (y + z) = (x + y) + z$;

for all $x, y, z \in \mathbb{R}^k$,

(3) there exists $\theta \in \mathbb{R}^k$ such that $x + \theta = x$ for all $x \in \mathbb{R}^k$;

(4) for every $x \in \mathbb{R}^k$ there exists $y \in \mathbb{R}^k$ such that $x + y = \theta$.

By using the above properties, it can be shown that the above k-tuples θ and y are uniquely determined. However, we can now at once see that $\theta = (0)_{i=1}^k$ and $y = (-x_i)_{i=1}^k$. In the sequel, these elements will be simply denoted by 0 and $-x$, respectively.

Now, for any $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^k$, we may also define

$$\lambda x = \big(\lambda x_i\big)_{i=1}^k .$$

Thus, it can be easily seen that the group $\mathbb{R}^k$ forms a vector space over $\mathbb{R}$ in the sense that, for all $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^k$, we have

(5) $1\,x = x$; (6) $\lambda(x + y) = \lambda x + \lambda y$;

(7) $(\lambda \mu)x = \lambda(\mu x)$; (8) $(\lambda + \mu)x = \lambda x + \mu x$.

Moreover, for any $x, y \in \mathbb{R}^k$, we may also define

$$\langle x, y \rangle = \sum_{i=1}^k x_i y_i .$$

Thus, it can be easily seen that $\langle\ \rangle$ is an inner product on $\mathbb{R}^k$ in the sense that, for all $\lambda \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^k$, we have

(9) $\langle x, x \rangle \geq 0$;

(10) $\langle\, x,\, y\,\rangle = \langle\, y,\, x\,\rangle\,;$ (11) $\langle\, x,\, x\,\rangle = 0$ implies $x = 0\,;$

(12) $\langle\, \lambda\, x,\, y\,\rangle = \lambda\, \langle\, x,\, y\,\rangle\,;$ (13) $\langle\, x + y\,,\, z\,\rangle = \langle\, x,\, z\,\rangle + \langle\, y,\, z\,\rangle\,.$

Now, by using the notation

$$\|\, x\,\| = \sqrt{\langle\, x,\, x\,\rangle}\,,$$

for all $\lambda,\, \mu \in \mathbb{R}$ and $x,\, y \in \mathbb{R}^k$, we can also state that

(14) $\|\, x\,\| \geq 0\,;$

(15) $\|\, \lambda\, x\,\| = |\, \lambda\,|\, \|\, x\,\|\,;$

(16) $\|\, x\,\| = 0$ implies $x = 0\,;$

(17) $\|\, \lambda\, x + \mu\, y\,\|^2 = \lambda^2\, \|\, x\,\|^2 + \mu^2\, \|\, y\,\|^2 + 2\, \lambda\, \mu\, \langle\, x,\, y\,\rangle\,.$

This polar identity (17) can also be written in the form

$$\|\, \lambda\, x + \mu\, y\,\|^2 = \big(\, \lambda\, \|\, x\,\| \,+\, \mu\, \|\, y\,\|\,\big)^2 + 2\, \lambda\, \mu\, \big(\, \langle\, x,\, y\,\rangle - \|\, x\,\|\, \|\, y\,\|\,\big)\,,$$

whose $\lambda = \mu = 1$ particular case also gives an instructive formula.

However, it is now more important to note that, by a particular case of (17), for any $r \in \mathbb{R}_+$ and $x,\, y \in \mathbb{R}^k$ we have

$$\|\, r\, x + y\,\|^2 = r^2\, \|\, x\,\|^2 \,+\, \|\, y\,\|^2 \,+\, 2\, r\, \langle\, x,\, y\,\rangle\,,$$

and thus

$$\frac{1}{r}\, \|\, r\, x \,+\, y\,\|^2 = r\, \|\, x\,\|^2 \,+\, \frac{1}{r}\, \|\, y\,\|^2 \,+\, 2\, \langle\, x,\, y\,\rangle\,.$$

Hence, by using the notations

$$\alpha\,(x,\, y) = \inf_{r \in \mathbb{R}_+} \frac{1}{r}\, \|\, r\, x + y\,\|^2 \qquad \text{and} \qquad \beta\,(x,\, y) = \inf_{r \in \mathbb{R}_+} \Big(\, r\, \|\, x\,\|^2 + \frac{1}{r}\, \|\, y\,\|^2\,\Big),$$

and a basic theorem on infimum, we can infer that

$$\alpha\,(x,\, y) = \beta\,(x,\, y) \,+\, 2\, \langle\, x,\, y\,\rangle\,.$$

In the next section, we shall show that

$$F\,(a,\, b) = \inf_{r \in \mathbb{R}_+} \big(\, a\, r + b/r\,\big) = \, 2\, \sqrt{a\, b}$$

for all $a,\, b \geq 0\,.$ Therefore, in particular, we can also state that

$$\beta\,(x,\, y) = \, 2\, \|\, x\,\|\, \|\, y\,\|\,,$$

and thus

$$\alpha(x, y) = 2\|x\|\|y\| + 2\langle x, y\rangle .$$

Hence, since $0 \le \alpha(x, y)$, we can already infer that

$$-\langle x, y\rangle \le \|x\|\|y\| .$$

Now, we can also easily see that

$$\langle x, y\rangle = -\langle -x, y\rangle \le \|-x\|\|y\| = \|x\|\|y\| .$$

Therefore, we can also state that

$$|\langle x, y\rangle| \le \|x\|\|y\| .$$

Now, by using this Schwarz inequality and the polar identity, we can also see that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = \big(\|x\| + \|y\|\big)^2,$$

and thus the triangle inequality

$$\|x+y\| \le \|x\| + \|y\|$$

also holds. Thus, in particular, $\|\ \|$ is a norm on $\mathbb{R}^k$.

In addition, we can also note that if $x, y \in \mathbb{R}^k$ such that $\|y\| \le 1$, then

$$\langle x, y\rangle \le |\langle x, y\rangle| \le \|x\|\|y\| \le \|x\| .$$

Moreover, if $x \ne 0$, then by taking $y = \lambda x$ with $\lambda = 1/\|x\|$, we can see that

$$\|\lambda x\| = |\lambda|\|x\| = 1 \quad \text{and} \quad \langle x, y\rangle = \langle x, \lambda x\rangle = \lambda\langle x, x\rangle = \lambda\|x\|^2 = \|x\| .$$

Therefore, we can also state that

$$\|x\| = \max_{\|y\|=1}\langle x, y\rangle = \max_{\|y\|\le 1}\langle x, y\rangle .$$

Note that the latter equalities also hold true for $x = 0$. Moreover, in the above equalities we may write $|\langle x, y\rangle|$ in place of $\langle x, y\rangle$.

Now, from the polar identity we can also see that

$$\max_{\|y\|=1}\|\lambda x + \mu y\|^2 = \max_{\|y\|=1}\Big(\lambda^2\|x\|^2 + \mu^2\|y\|^2 + 2\lambda\mu\langle x, y\rangle\Big)$$

$$= \lambda^2 \| x \|^2 + \mu^2 + 2 \lambda \mu \| x \| = \big(\lambda \|x\| + \mu \big)^2 ,$$

and thus

$$\max_{\| y \|=1} \| \lambda x + \mu y \| = \big| \lambda \|x\| + \mu \big|$$

for all $\lambda, \mu \in \mathbb{R}$ and $x \in \mathbb{R}^k$.

2. The determination of the value $F_{\mathbb{R}_+}(a, b)$

Now, because of our former observations, we may naturally consider the following

Problem 2.1. *For all* $a, b \in \mathbb{R}$, *determine*

$$F(a, b) = \inf_{r \in \mathbb{R}_+} \Big(a r + \frac{b}{r} \Big) .$$

Remark 2.2. By defining

$$f(r) = a r + b/r$$

for all $r \in \mathbb{R}_+$, we can state that

$$F(a, b) = \inf_{r \in \mathbb{R}_+} f(r) .$$

Now, if $\boxed{a < 0}$, then we can note that

$$\lim_{r \to +\infty} f(r) = \lim_{r \to +\infty} \big(a r + b/r \big) = -\infty .$$

Therefore, we have

$$F(a, b) = \inf_{r \in \mathbb{R}_+} f(r) = -\infty .$$

While, if $\boxed{b < 0}$, then we can note that

$$\lim_{r \to 0} f(r) = \lim_{0 < r \to 0} \big(a r + b/r \big) = -\infty .$$

Therefore, we again have

$$F(a, b) = \inf_{r \in \mathbb{R}_+} f(r) = -\infty .$$

On the other hand, if $\boxed{a = 0 \leq b}$, then we can note that

$$f(r) = b/r \geq 0 \quad \text{for all} \quad r \in \mathbb{R}_+ \qquad \text{and} \qquad \lim_{r \to +\infty} f(r) = \lim_{r \to +\infty} b/r = 0\,.$$

Therefore, we have

$$F(a,\, b) = \inf_{r \in \mathbb{R}_+} f(r) = 0\,.$$

While, if $\boxed{b = 0 \leq a}$, then we can note that

$$f(r) = a\,r \geq 0 \quad \text{for all} \quad r \in \mathbb{R}_+ \qquad \text{and} \qquad \lim_{0 < r \to 0} f(r) = \lim_{0 < r \to 0} a\,r = 0\,.$$

Therefore, we again have

$$F(a,\, b) = \inf_{r \in \mathbb{R}_+} f(r) = 0\,.$$

Now, by summarizing our former observations, we can state that

$$F(a,\, b) = \begin{cases} -\infty & \text{if} \quad a < 0 \quad \text{or} \quad b < 0\,, \\ \quad 0 & \text{if} \quad a = 0 \leq b \quad \text{or} \quad b = 0 \leq a\,. \end{cases}$$

The above Remark 2.2 shows that, to solve Problem 2.1, we need actually solve

Problem 2.3. *For all* $a,\, b \in \mathbb{R}_+$, *determine*

$$F(a,\, b) = \inf_{r \in \mathbb{R}_+} \left(a\,r + \frac{b}{r} \right).$$

In the sequel, we shall give four different solutions for this simplified problem. The first and second ones have been suggested by Gy. Maksa and P. Pasteczka, respectively.

Solution 2.4. By the arithmetic-geometric mean inequality, we can see that

$$2\sqrt{a\,b} = 2\sqrt{(ar)\,(b/r)} \leq a\,r + b/r = f(r)$$

for all $r \in \mathbb{R}_+$.

Moreover, by taking

$$u = \sqrt{b/a}\,,$$

we can see that

$$f(u) = au + b/u = a\sqrt{b/a} + b\sqrt{a/b} = 2\sqrt{a\,b}\,.$$

Hence, it is clear that

$$F(a,\,b) = \inf_{r\in\mathbb{R}_+} f(r) = \min_{r\in\mathbb{R}_+} f(r) = f(u) = 2\sqrt{a\,b}\,.$$

Solution 2.5. The use of the arithmetic-geometric mean inequality can be avoided by noticing that

$$f(r) = a\,r + b/r = a\,r - 2\sqrt{a\,b} + b/r + 2\sqrt{a\,b} = \left(\sqrt{a\,r} - \sqrt{b/r}\right)^2 + 2\sqrt{a\,b}$$

for all $x \in \mathbb{R}_+$.

Hence, it is clear that the smallest value of $f(r)$ is attained whenever r is such that $\sqrt{a\,r} - \sqrt{b/r} = 0$. That is, $r = \sqrt{b/a} = u$. Thus, the conclusion of Solution 2.4 can again be stated.

Solution 2.6. We can note that the function f, considered in Remark 2.2, is differentiable and

$$f'(r) = a - b/r^2$$

for all $r \in \mathbb{R}_+$.

Therefore, for any $r \in \mathbb{R}_+$, we have

$$f'(r) \le 0 \iff a - b/r^2 \le 0 \iff r \le \sqrt{b/a} \iff r \le u$$

and

$$0 \le f'(r) \iff 0 \le a - b/r^2 \iff \sqrt{b/a} \le r \iff u \le r.$$

Hence, by a basic theorem of calculus, we can see that f is decreasing on $]0,\,u]$ and f is increasing on $[\,u,\,\infty\,[$. Thus, the conclusion of Solution 2.4 can again be stated.

Solution 2.7. The use of the corresponding results of differential calculus can be avoided, by defining

$$\Delta_h(r) = f(r+h) - f(r)$$

for all $r, h \in \mathbb{R}_+$.

Namely, thus for any $r, h \in \mathbb{R}_+$ we have

$$f(r) \geq f(r+h) \iff \Delta_h(r) \leq 0 \quad \text{and} \quad f(r) \leq f(r+h) \iff 0 \leq \Delta_h(r) .$$

Moreover, we can note that

$$\Delta_h(r) = f(r+h) - f(r) = a(r+h) + \frac{b}{r+h} - \left(a\,r + \frac{b}{r} \right) = a\,h - b\left(\frac{1}{r} - \frac{1}{r+h} \right)$$
$$= a\,h - \frac{b\,h}{r(r+h)} = a\,h \left(1 - \frac{b/a}{r(r+h)} \right) = a\,h \left(1 - \frac{u^2}{r(x+h)} \right) .$$

Therefore, we have

$$\Delta_h(r) \leq 0 \iff 1 - \frac{u^2}{r(r+h)} \leq 0 \iff r(r+h) \leq u^2$$

and

$$0 \leq \Delta_h(r) \iff 0 \leq 1 - \frac{u^2}{r(r+h)} \iff u^2 \leq r(r+h) .$$

Hence, it is clear that

$$f(r) \geq f(r+h) \quad \text{if} \quad r+h \leq u \qquad \text{and} \qquad f(r) \leq f(r+h) \quad \text{if} \quad u \leq r .$$

Now, since for any $r, s \in \mathbb{R}_+$ with $r < s$ we have $s = r + h$ with $h = s - r > 0$, we can again infer that f is decreasing on $]0, u]$ and f is increasing on $[u, \infty[$. Thus, the conclusion of Solution 2.4 can again be stated.

The above four solutions for Problem 2.3 differ in difficulty and prerequisites. The most simple one is Solution 2.5. However, each of them can be used for some teaching purposes in secondary and high schools.

Remark 2.8. Now, by Remark 2.2 and Solution 2.4, as a solution to Problem 2.1 we can state that

$$F(a, b) = \begin{cases} -\infty & \text{if} \quad a < 0 \quad \text{or} \quad b < 0 , \\ 2\sqrt{a\,b} & \text{if} \quad a \geq 0 \quad \text{and} \quad b \geq 0 . \end{cases}$$

3. The determination of the value $F_{A_+}(a, b)$ for a dense subset A of $\mathbb{R}$

For any $x, y \in \mathbb{R}^k$, we may also naturally define

$$d(x, y) = \| x - y \| .$$

Thus, it can be easily seen that d is a metric on $\mathbb{R}^k$ in the sense that, for any $x, y, z \in \mathbb{R}^k$, we have

(1) $d(x, x) = 0$;

(2) $d(x, y) = d(x, y)$;

(3) $d(x, y) = 0$ implies $x = y$;

(4) $d(x, z) \leq d(x, y) + d(y, z)$.

Moreover, d has the useful additional properties that, for any $\lambda \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^k$, we have:

(5) $d(\lambda x, \lambda y) = |\lambda| \, d(x, y)$; (6) $d(x + z, y + z) = d(x, y)$.

The translation invariance (6) is a surprisingly strong property. Namely, together with the triangle inequality (4), it already implies that

$$d(x + y, z + w) \leq d(x + y, z + y) + d(z + y, z + w) = d(x, z) + d(y, w)$$

for all $x, y, z, w \in \mathbb{R}^k$. (Show that the corresponding inequality also implies (6).)

Moreover, by using properties (6) and (2), we can also easily see that

$$d(-x, -y) = d(0, x - y) = d(y, x) = d(x, y)$$

for all $x, y \in \mathbb{R}^k$. However, the latter fact is a more immediate consequence of the homogeneity property (5).

Now, for any point x and sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$, we may naturally write

$$x \in \lim_{n\to\infty} x_n \qquad \text{if} \qquad \overline{\lim_{n\to\infty}} \, d(x, x_n) = 0$$

and $x \in \operatorname*{adh}_{n\to\infty} x_n$ if $\underline{\lim}_{n\to\infty} d(x, x_n) = 0$.

Recall that

$$\overline{\lim_{n\to\infty}}\ d(x,\, x_n) = \inf_{n\in\mathbb{N}}\ \sup_{k\geq n}\ d(x,\, x_k)\,.$$

Therefore, $x \in \lim_{n\to\infty} x_n$ if and only if for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, with $k \geq n$, we have $d(x,\, x_k) < \varepsilon$.

Hence, by using the triangle inequality, it can be easily seen that for any sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$ the set $\lim_{n\to\infty} x_n$ is either empty or a singleton. Therefore, we may simply write $x = \lim_{n\to\infty} x_n$ instead of $x \in \lim_{n\to\infty} x_n$.

The corresponding assertion is not true for the adherence. For instance, we can easily see that $\operatorname{adh}_{n\to\infty} (-1)^n = \{-1,\, 1\}$. Namely, it can be shown that for some $x \in \mathbb{R}^k$ and sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$, we have $x \in \operatorname{adh}_{n\to\infty} x_n$ if and only if there exists a subsequence $(x_{k_n})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x = \lim_{n\to\infty} x_{k_n}$.

Now, for any $A,\, B \subseteq \mathbb{R}^k$, we may also naturally define

$$\underline{d}\big(A,\, B\big) = \inf\big\{\, d(a,\, b): \quad a \in A\,, \quad b \in B \,\big\}$$

and $\overline{d}\big(A,\, B\big) = \sup\{\, d(a,\, b): \quad a \in A\,, \quad b \in B\,\}$.

Thus, $\underline{d}$ and $\overline{d}$ are only some useful distance functions on the family of all subsets of $\mathbb{R}^k$. By using them, for any $x \in X$ and $A \subseteq \mathbb{R}^k$, we may naturally define

$$\underline{d}(x,\, A) = \underline{d}\,\big(\{x\},\, A\big)$$

and $d(A) = \operatorname{diam}(A) = \overline{d}\,(A,\, A)$.

Moreover, if $A,\, B \subseteq \mathbb{R}^k$, then by noticing that $\underline{d}\,\big(A,\, B\big) = \inf_{x\in A}\ \underline{d}\,(x\,,\, B)$, we may also naturally define

$$E\,(A,\, B) = \sup_{x\in A}\ \underline{d}\,(x\,,\, B)$$

and $D\,(A,\, B) = \max\big\{\, E\,(A,\, B)\,,\ E\,(B,\, A)\big\}$. (Show that the excess E and the Hausdorff distance D have some better properties than the gap $\underline{d}$.)

However, it is now more important to note that, for any $A \subseteq \mathbb{R}^k$, we may also naturally define

$$A^{-} = \operatorname{cl}(A) = \big\{\, x \in X: \quad \underline{d}\,(x,\, A) = 0\,\big\}\,,$$

and $A^{\circ} = \operatorname{int}(A) = \mathbb{R}^k \setminus \big(\mathbb{R}^k \setminus A\big)^{-}$.

Now, for instance, a subset A of $\mathbb{R}^k$ may be naturally called closed if $A^{-} \subseteq A$, and open if $A \subseteq A^{\circ}$. Moreover, for instance, A may be naturally called dense if $A^- = \mathbb{R}^k$, and fat if $A^\circ \neq \emptyset$. Thus, under the notation $A^c = \mathbb{R}^k \setminus A$, it can be easily shown that A is open if and only if A^c is closed, and A is fat if and only if A^c is not dense.

By using the above basic definitions, it can be easily seen that for any $x \in \mathbb{R}^k$ and $A \subseteq \mathbb{R}^k$, we have $x \in A^-$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x = \lim_{n\to\infty} x_n$. Thus, in particular A is a dense subset of $\mathbb{R}^k$ if and only if for each $x \in \mathbb{R}^k$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x = \lim_{n\to\infty} x_n$.

By using the integral part function and the irrationality of $\sqrt{2}$ (see Boros & Száz (1998, 1999)), it can be easily shown that both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$. Hence, by using an obvious reduction of the convergence in $\mathbb{R}^k$ to that in $\mathbb{R}$, it is clear that both $\mathbb{Q}^k$ and $\big(\mathbb{R}\setminus\mathbb{Q}\big)^k$ are also dense in $\mathbb{R}^k$.

Now, as a substantial generalization of Problem 2.1, we may also naturally consider the following

Problem 3.1. *For all* $a, b \in \mathbb{R}$ *and dense subset* A *of* $\mathbb{R}$, *determine*

$$F(a, b) = \inf_{r\in A_+}\Big(a\,r + \frac{b}{r}\Big).$$

Solution 3.2. Since $A \subseteq \mathbb{R}$, and thus $A_+ \subseteq \mathbb{R}_+$, it is clear that

$$\big\{ f(r): \quad r \in A_+ \big\} \subseteq \big\{ f(x): \quad r \in \mathbb{R}_+ \big\}.$$

Hence, by using a basic property of the infimum, we can infer that

$$F_{\mathbb{R}_+}(a, b) = \inf \big\{ f(r): \quad r \in \mathbb{R}_+ \big\} \leq \inf \big\{ f(r): \quad r \in A_+ \big\} = F_{A_+}(a, b).$$

On the other hand, if $\varepsilon > 0$, then by the definition of infimum we can see that there exists $r \in \mathbb{R}_+$ such that

$$f(r) < F_{\mathbb{R}_+}(a, b) + \varepsilon.$$

Moreover, since A is dense in $\mathbb{R}$, there exists a sequence $(r_n)_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} r_n = r$. Hence, since $0 < r$, it is clear that there exists $k \in \mathbb{N}$ such that $0 < r_n$, and thus $r_n \in A_+$ for all $n \in N$ with $n \geq k$. Moreover, by some basic theorems on limits, we can see that

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty}\big(a\,r_n + b/r_n \big) = a\,r + b/r = f(r) < F_{\mathbb{R}_+}(a, b) + \varepsilon.$$

Thus, in particular, there exists $n \in \mathbb{N}$ with $n \geq k$ such that

$$f(r_n) < F_{\mathbb{R}_+}(a, b) + \varepsilon .$$

Hence, since $r_n \in A_+$, we can infer that

$$F_{A_+}(a, b) = \inf_{r \in A_+} f(r) \leq f(r_n) < F_{\mathbb{R}_+}(a, b) + \varepsilon .$$

Now, by taking $\lim_{0<\varepsilon \to 0}$, we can also state that

$$F_{A_+}(a, b) \leq F_{\mathbb{R}_+}(a, b) ,$$

and thus

$$F(a, b) = F_{A_+}(a, b) = F_{\mathbb{R}_+}(a, b) = \begin{cases} -\infty & \text{if} \quad a < 0 \quad \text{or} \quad b < 0 , \\ 2\sqrt{ab} & \text{if} \quad a \geq 0 \quad \text{and} \quad b \geq 0 . \end{cases}$$

Remark 3.3. Note that in the latter equality, in particular, we may write both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in place of A.

4. The determination of the value $F_{\mathbb{N}}(a, b)$

In this section, motivated by the above investigations, we may also naturally consider the following

Problem 4.1. *For all* $a, b \in \mathbb{R}$, *determine*

$$F(a, b) = \inf_{n \in N} \left(a\, n + \frac{b}{n} \right) .$$

Remark 4.2. If $\boxed{a < 0}$, then from Remark 2.2 we can see that

$$\lim_{n \to \infty} f(n) = \lim_{r \to +\infty} f(r) = -\infty .$$

Therefore,

$$F(a, b) = \inf_{n \in \mathbb{N}} f(n) = -\infty .$$

Moreover, if $\boxed{a = 0 \le b}$, then from Remark 2.2 we can see that

$$f(n) = b/n \ge 0 \quad \text{for all} \;\; n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to\infty} f(n) = \lim_{r\to+\infty} f(r) = 0\,.$$

Therefore,

$$F(a, b) = \inf_{n\in\mathbb{N}} f(n) = 0.$$

While, if $\boxed{b \le 0 \le a}$, then from Solution 2.7 we can see that

$$f(n+1) - f(n) = \Delta_1(n) = a - \frac{b}{n(n+1)} \ge 0$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\big(f(n)\big)_{m=1}^{\infty}$ is increasing, and thus

$$F(a, b) = \inf_{n\in\mathbb{N}} f(n) = \min_{n\in\mathbb{N}} f(n) = f(1) = a + b\,.$$

Now, by summarizing our former observations, we can state that

$$F(a, b) = \begin{cases} -\infty & \text{if} \quad a < 0\,, \\ 0 & \text{if} \quad a = 0 \le b\,, \\ a + b & \text{if} \quad b \le 0 \le a\,. \end{cases}$$

The above Remark 4.2 shows that, to solve Problem 4.1, we need actually solve

PROBLEM 4.3. *For all* $a, b \in \mathbb{R}_+$, *determine*

$$F(a, b) = \inf_{n\in\mathbb{N}} \Big(a\,n + \frac{b}{n} \Big)\,.$$

In the sequel, by using a similar argument as in Solutions 2.6 and 2.7, we shall give two different solutions for this simplified problem.

SOLUTION 4.4. From Solution 2.6, we can see that f is decreasing on $]0, u]$ and f is increasing on $[u, \infty[$ where $u = \sqrt{b/a}$.

Hence, if $\boxed{b \le a}$, i.e., $u \le 1$, we can see that the sequence $\big(f(n)\big)_{n=1}^{\infty}$ is increasing, and thus

$$F(a, b) = \inf_{n\in\mathbb{N}} f(n) = \min_{n\in\mathbb{N}} f(n) = f(1) = a + b\,.$$

While, if $\boxed{a < b}$, i. e., $1 < u$, then by defining

$$k = [\,u\,] = \max\{\, n \in \mathbb{N} : \quad n \leq u \,\}$$

we can see that $k \leq u < k+1$, and thus

$$F(a, b) = \inf_{n \in \mathbb{N}} f(n) = \min\big\{\, f(k),\ f(k+1) \big\}\,.$$

Moreover, by using that

$$f(k+1) - f(k) = \Delta_1(k) = a - b/k(k+1)\,,$$

we can note that

$$f(k) \leq f(k+1) \iff 0 \leq a - b/k(k+1) \iff u^2 \leq k\,(k+1)$$

and

$$f(k+1) \leq f(k) \iff a - b/k(k+1) \leq 0 \iff k\,(k+1) \leq u^2.$$

Therefore,

$$F(a, b) = f(k) = a\,k + b/k \qquad \text{if} \qquad u^2 \leq k\,\big(\,k+1\big)$$

and

$$F(a, b) = f(k+1) = a\,(k+1) + b/(k+1) \qquad \text{if} \qquad k\,(k+1) \leq u^2.$$

Now, by summarizing our former observations we can state that

$$F(a, b) = \begin{cases} a + b & \text{if} \quad b \leq a\,, \\ a\,[\,u\,] + b/[\,u\,] & \text{if} \quad a < b\,, \quad u^2 \leq [\,u\,]\big([\,u\,]+1\big)\,, \\ a\big([\,u\,]+1\big) + b/\big([\,u\,]+1\big) & \text{if} \quad a < b\,, \quad [\,u\,]\big([\,u\,]+1\big) \leq u^2. \end{cases}$$

Remark 4.5. Moreover, by Remark 4.2 and Solution 4.4, as a solution to Problem 4.1 we can state that

$$F(a, b) = \begin{cases} -\infty & \text{if} \quad a < 0\,, \\ 0 & \text{if} \quad a = 0 \leq b\,, \\ a + b & \text{if} \quad b \leq 0 \leq a \quad \text{or} \quad 0 < b \leq a\,, \\ a\,[\,u\,] + b/[\,u\,] & \text{if} \quad 0 < a < b\,, \quad u^2 \leq [u]\big([u]+1\big)\,, \\ a\big([\,u\,]+1\big) + b/\big([\,u\,]+1\big) & \text{if} \quad 0 < a < b\,, \quad [u]\big([u]+1\big) \leq u^2. \end{cases}$$

SOLUTION 4.6. If $\boxed{b \leq 2\,a}$, then we can see that

$$0 \leq a\,2 - b \leq a\,n\,(n+1) - b\,,$$

and thus

$$0 \leq a - b/n(n+1) = \Delta_1(n) = f\,(n+1) - f\,(n)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\big(f\,(n)\big)_{n=1}^{\infty}$ is increasing, and thus

$$F\,(a,\,b) = \inf_{n\in\mathbb{N}} f\,(n) = \min_{n\in\mathbb{N}} f\,(n) = f(1) = a + b\,.$$

Therefore, in the sequel, we need only consider the case $\boxed{2\,a < b}$. For this, we can first note that

$$\lim_{n\to\infty} \Delta_1(n) = \lim_{n\to\infty} \big(a - b/n(n+1)\big) = a > 0\,.$$

Therefore, $\Delta_1(n) > 0$ for all sufficiently large natural numbers n. Thus, in particular

$$m = \min\big\{\, n \in \mathbb{N} : \quad \Delta_1(n) \geq 0 \,\big\}$$

exists. Moreover, since $\Delta_1(1) = a - b/2 < 0$, we can state that $m > 1$.

Furthermore, since $b > 0$, we can also note that

$$\Delta_1(n) = a - b/n(n+1) < a - b/(n+1)(n+2) = \Delta_1(n+1)$$

for all $n \in \mathbb{N}$. Thus, the sequence $\big(\Delta_1(n)\big)_{n=1}^{\infty}$ is increasing.

Therefore, for any $n \in \mathbb{N}$, we can state that

$$\Delta_1(n) < 0 \quad \text{if} \quad n < m \qquad \text{and} \qquad 0 \leq \Delta_1(n) \quad \text{if} \quad n \geq m\,,$$

and thus

$$f\,(n) > f\,(n+1) \quad \text{if} \quad n < m \qquad \text{and} \qquad f\,(n) \leq f\,(n+1) \quad \text{if} \quad n \geq m\,.$$

Hence, it is clear that

$$f\,(1) > f\,(2) > \ \ldots\ > f\,(m-1) > f\,(m) \leq f\,(m+1) \leq f\,(m+2) \leq \ \ldots\,,$$

and thus

$$F\,(a,\,b) = \inf_{n\in\mathbb{N}} f\,(n) = \min_{n\in\mathbb{N}} f\,(n) = f(m) = a\,m + b/m\,.$$

Now, to determine m, we can also note that

$$\begin{aligned} 0 \leq \Delta_1(n) \iff 0 \leq a - b/n(n+1) \iff 0 \leq a\left(n^2+n\right) - b \\ \iff b/a \leq n^2 + n \iff b/a + 1/4 \leq \left(n + 1/2\right)^2 \\ \iff \sqrt{b/a + 1/4} \leq n + 1/2 \iff -1/2 + \sqrt{b/a+1/4} \leq n \,. \end{aligned}$$

Hence, it is clear that, under the notation

$$\lambda = -1/2 + \sqrt{b/a + 1/4}\,,$$

we have

$$m = \min\left\{ n \in \mathbb{N} : \quad 0 \leq \Delta_1(n) \right\} = \min\left\{ n \in \mathbb{N} : \quad \lambda \leq n \right\}.$$

Now, by summarizing our former observations, we can state that

$$F(a, b) = \begin{cases} a + b & \text{if} \quad b \leq 2a\,, \\ am + b/m & \text{if} \quad 2a < b\,. \end{cases}$$

Remark 4.7. Moreover, by Remark 4.2 and Solution 4.6, as a solution to Problem 4.1 we can state that

$$F(a, b) = \begin{cases} -\infty & \text{if} \quad a < 0\,, \\ 0 & \text{if} \quad a = 0 \leq b\,, \\ a + b & \text{if} \quad 0 \leq a \quad \text{and} \quad b \leq 2a\,, \\ am + b/m & \text{if} \quad 0 < a \quad \text{and} \quad 2a < b\,, \end{cases}$$

where

$$m = \min\left\{ n \in \mathbb{N} : \quad \lambda \leq n \right\} \qquad \text{with} \qquad \lambda = -1/2 + \sqrt{b/a + 1/4}\,.$$

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