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Teaching Mathematics and Computer Science

"On the way" to the function concept – experiences of a teaching experiment

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Abstract. Knowing, comprehending and applying the function concept is essential not only from the aspect of dealing with mathematics but with several scientific fields such as engineering. Since most mathematical notions cannot be acquired in one step (Vinner, 1983) the development of the function concept is a long process, either. One of the goals of the process is evolving an "ideal" concept image (the image is interrelated with the definition of the concept). Such concept image plays an important role in solving problems of engineering. This study reports on the beginning of a research aiming the scholastic forming of the students' function concept image i.e. on the experiences of a "pilot" study. By the experiment, we are looking for the answer of the following question: how can the analysis of such function relations be built into the studied period (8th grade) of the evolving process of the function concept that students meet in everyday life and also in engineering life?

Key words and phrases: formation of the function concept, concept image, real life situations, engineering area.

ZDM Subject Classification: D43, U73.

Introduction

"Mathematics is such a severe specialty that you should not fail any occasion to make it more entertaining." (Blaise Pascal)

Similarly to most mathematical notions, function concept is also very abstract. In the Central European countries in the teaching-learning process of the function concept this abstract mathematical character of the concept is dominated with very few applications and modelling tasks. But to develop the concept, a number of researches used real life situations and several representations of covariant quantities hiding in the context (Sierpinska, 1992; Davidenko, 1997). The tasks taken from real life ensure that the students– activating their real life experiences–invest themselves in the given situation and use their experiences and knowledge for the solution (Kelemen, 2010).

Contrary to elementary notions (e.g. the point) having ostensible definition (i.e. when defining it, we show a point saying ”it is a point”) (Blaskó & Hamp, 2007), the function has a formal definition. Though, in many cases, students do not use a definition of the given notion for deciding whether a mathematical object is an example or a counter-example for the notion. They majorly decide based on the concept image. Concept image is a set of all mental ideas that a person associate with in his mind, together with its attributes, when hearing the name of the given concept (Vinner & Dreyfus, 1989; Vinner, 1992). The development of students' concept images are largely impressed by observations, visual representations, experiences and concrete examples etc. gained in the evolving process of the concept (Ambrus, 2003). The concept image closely related to the application of the definition of the concept and in many cases it is necessary to control the concept image with the definition (Vinner, 1983).

There are several researches about that students entering higher education have a function concept not meeting the requirements of the curriculum framework, the function concept is highly difficult for them (Vinner, 1983; Szanyi, 2015, 2017b). Students think about functions as if they are surely able to be defined by a formula (by a single analytic formula) (Sierpinska, 1992; Carlson, 1998; Clement, 2001), so the understanding of an expression with split domain very difficult for them (Carlson, 1998; Vinner & Dreyfus, 1989). In their imagination, the only existing functions are those which they met in the school and used in general, which are able to be defined by a rule of assignment and having type $\mathbb{R} \to \mathbb{R}$ from the set of real numbers to the set of real numbers). Similar concept image could be observed in the case of 8th grade elementary school students, also (Szanyi, 2017b).

However, students entering into engineering education meet even vector-valued functions (of type $\mathbb{R} \to \mathbb{R}^2$) already in the beginning of their higher studies. Such functions are widely applied in several engineering fields (e.g. in the case of throwing an object at an angle) (Figure 1). This is the reason why it is important for the students to understand functions properly. However, by our collective or personal experiences it can be seen that these functions are far abstract for most students. In our opinion, one of the main reasons for it is that the essence of the concept itself "has been lost" during high school studies because of the specific functions (Szanyi, 2017a, 2017b).

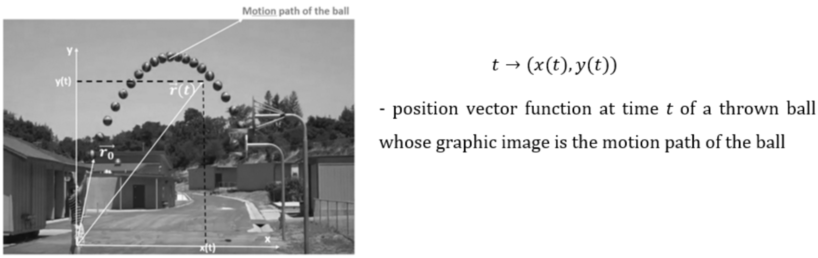


Figure 1. Motion path of the ball

Our study presents the beginning of a main research (pilot study). The results of a pilot study can lead to a research of greater volume so it foregoes the main research, gives guidance and ideas for the precise planning of that (Forgasz & Kaur, 1997). The main goal of the research: (1) scholar development of the students' function concept by examining the connection between quantities of everyday life and those of engineering; (2) let the students sense long before choosing their career how essential the thorough knowledge and understanding of the several representations of functions are in everyday life and mainly in engineering life. In the main research we study the following question: *can the examination of function relationships in everyday life and engineering life helps develop the "ideal" concept image?*

Educational aspects of the development of the function concept

Function is one of the basic concept of mathematics which many kinds of definitions and representations are known of. This can be one reason for why students have difficulties in understanding it properly. The scholastic forming of functions is highly effected by its historical development (Kleiner, 1989): in the course of its preparation, the connections characterized by rules are discussed (Herendine, 2013), then–following historical development–the definitions miss describing relationships exclusively by formulas. Vinner (1992) called the modern function definition Dirichlet-Bourbaki definition which has a dominant characteristic in the case of teaching the function concept: *"The function is a correspondence between two non-empty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the range)."* (p. 357)

By Sajka (2003), it is not only the wide variety of its definitions and representations that cause the students' difficulties in understanding the function concept but also its twofold nature since it can be interpreted in two, basically different ways: structurally–as an object and operatively–as a process. And these ways have to constitute a coherent unity (Sfard, 1991). Consequently, the elementary *procept* (process+concept) is a part of the developing process of the function concept that has three components: a *process* ("input-transformation-output"), a mathematical object (*concept*) that is resulted by the process and a *symbol* representing the process or the object, either (representation of the function formula) (Gray & Tall, 1994). DeMarois and Tall (1996) consider the procept the "innermost" layer of the function concept which can be reached when a student goes along all the four stages (pre-function, action, process, object) of understanding the concept (Dubinsky & Harel, 1992).

The three components of the procept have an important role in understanding the function concept since they include the constitutive parts of the concept (variables, relationships, rules) (Figure 2) (own editing).

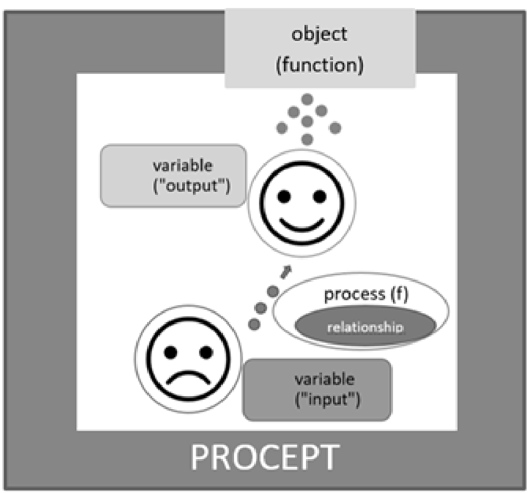


Figure 2

The constitutive parts of our concept can be found in several "worlds" whose cognition is considered the assumption of understanding of the function concept, by Sierpinska (1992):

- world of change or changing objects,
- world of relationships or processes,
- world of rules, patterns, laws.

So overall, it can be stated that–in order to understand the concept–the students need to know these "worlds" mentioned above which, including the constitutive parts of the concept, and basics of the elementary procept. Besides all these, an important aspect of understanding the concept is the ability to apply several representations and translate the needed qualities from a representation to another (Lin & Cooney, 2001, cited by Areti et al., 2015, p. 140). Parallelly with introducing, we have to put in great effort the students' knowing the connection between distinct representations when dealing with them. If it is not fulfilled during the development of the concept, then these representations build into their knowledge isolatedly from each other (Thompson, 1994).

According to Sierpinska (1992), when teaching the function concept, it is expedient to motivate to know the concept by real life problems so that students can sense and verbalize the changes and identify the connection between variables. The ability to sense changes, to recognize the correspondence between them and to represent the connection by a formula is essential even by the aspect of studying the field of engineering. In order to solve the problems arisen, it is often needful to find the proper mathematical model which most frequently the function concept can help with (Selden & Selden, 1992).

Besides all these, in the process of the development of the concept, it is important to specify the students' level of understanding the concept. There were several studies dealing with specifying these levels (Bergeron & Herscovics, 1982). By Vollrath (1984, cited by Ambrus, 2004, p. 66), a student acquires a concept if he/she possesses the following verifiable abilities:

(1) He/she can give a definition of the concept.

(2) He/she can decide whether a given object is related to the concept (concept identification).

(3) He/she can give (construct) examples of the concept (concept realization).

(4) He/she knows the properties of the concept.

(5) He/she can use the concept or its properties to describe situations or solve problems.

(6) He/she can arrange the concept into the hierarchical system of concepts.

However, it is a long way till possessing these abilities. Piaget (1955, cited by Dienes, 2015) emphasizes that in the long process of acquiring a concept it is essential that the students play with the elements of the concept (in proper time and with games matching to the given age group) and discover the concept inductively. This was the principle of Zoltán Dienes, too. In his opinion, if mathematics is difficult for someone, its reason could be that he/she can't relate the concept to some kind of object, activity, experience (Czegledi, 2011).

However, if sufficiently enough empirical material is given to the children, then they can acquire the function concept such effectively than the concepts of the variable or the number (Dienes, 2015).

The following occupations aimed at developing the function concept with the help of activities built upon each other so that it gives as much experimental material to the students as it is possible about the concept, by discovering and studying the "worlds" mentioned above in everyday processes.

Methodology and experiences of the "pilot study"

Background and methodology

In our teaching experiment, 12 students from the same class of Nagydobrony Secondary School took part who attend the 8th grade, non-special mathematical department and study Mathematics four hours per week. Based on the half-year marks, their general mathematical knowledge can be considered average. The teaching experiment was in May 22, 2018, within the frames of three consecutive, 45-minute lessons. In the case of the students, the directed preparation of the function concept started only in the 6th grade (this preparation wasn't emphasized in the curriculum framework of the lower school) (Szanyi, 2017a). Then the students took part in an experimental teaching series which six class works were involved in. This was the first time when students got to know the constitutive parts of the function concept and its several representations (Szanyi, 2017a). The results of this experiment were the following: the sources of the function (function machine, process taken from real life) facilitated the familiarization with the constitutive parts of function; the formation of the concept of the contextually embedded covariant quantities and the recognition of the relationship between them - thanks to the recognition of different function representations - was successful. The students had met the definition of the function for the first time in the spring term of the 7th grade. In this grade were familiarized with the linear function. Prior to our pilot study of teaching, in the fall semester of the 8th grade, the issue of functions were discussed again based on the curriculum framework: they dealt with the function definition again and also with some further specific functions (quadratic function, square root function, rational function).

The tasks of the occupations were done in groups, whereas setting the tasks and the discussion of the experience took place in the whole class. We applied co-operative teaching method, because it is suitable for these activity-based lessons: students have the opportunity to try explain and control their ideas in a small group of their classmates. They were familiar with this way of learning. The children were divided into 2 groups, both groups had 6 members. All parts of the lesson were documented by voice recorder, photos and we collected the notes of the students.

Description of the experiment and results

Revitalization of the function concept

The aim of this part of our experiment was to revitalize the function concept and its different representations on the inductive way.

Since in the chosen student group (aged 13-14 years) the function concept had already been introduced previously, in order to recall it, we invited the students to have some teamwork game. Using several representations of the relationships (verbal, table, arrow diagram, graph, formula), we asked both groups of 6 persons to analyse the relationships between several sets (among them we gave connections often occurred in engineering calculations also e.g. "number to number", "number to number pair" assignment etc.). One student of each group was responsible for one relationship (exercise). In each exercises, the relationship s were represented diversely and the student responsible for that task had to represent the relationship differently from the given form and he/she had to name the covariant quantities (elements) (Table 1).

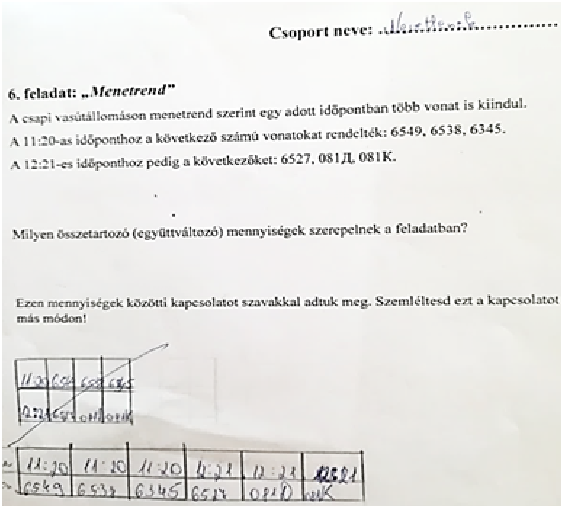
Number of exercise	Relationship	Representational form
1	relationship between the number of bought booklets and the amount paid for them	table
2	relationship between the letters of alphabet and the student having a surname beginning with the given letter	verbally (in words)
3	relationship between elapsed time and the speed of the car	graph
4	relationship between real numbers beside a divided set (distinct rule of assignment on distinct sets)	formula
5	relationship between the sides and the area of a rectangle	arrow diagram
6	relationship between times and leaving trains	verbally (in words)

Table 1

In the course of observing students' work, it could be detected well that the continuous development of phrasing the relationship in words (verbal representation) i.e. students' relational vocabulary wasn't emphasized enough. When phrasing the relationship of covariant quantities in words, they could hardly recognize the covariant quantities (exercises 2 and 6) and change into another representation (Figure 3[1]).

Students needed some induction so that distinct representations of covariant quantities could come into their mind. Then we called their attention to look at their classmates' exercises: how connections between covariant quantities are shown there and which of them could be used in the exercises of their own to represent this relationship. The most popular representation was the table. The reason for it is that when the function concept was introduced in the 7th grade, the participating students' textbook on Mathematics prefers this representation for illustrating the relationship between covariant quantities, later the concept is represented by a formula and for making the chart, a value table is constructed

[1] ***Exercise 6: Schedule***
In the railway station of Csap, there are several trains leaving at a given time by the schedule.
To the time 11:20, the following trains are assigned: 6549, 6538, 6345.
To the time 12:21, the following ones are assigned to: 6527, 081D, 081K.
What covariant quantities are in the exercise?
The relationship between these quantities was given in words.
Represent this relationship in another form.


Csoport neve: ....

6. feladat: „*Menetrend*"

A csapi vasútállomáson menetrend szerint egy adott időpontban több vonat is kiindul.
A 11:20-as időponthoz a következő számú vonatokat rendelték: 6549, 6538, 6345.
A 12:21-es időponthoz pedig a következőket: 6527, 081Д, 081K.

Milyen összetartozó (együttváltozó) mennyiségek szerepelnek a feladatban?

Ezen mennyiségek közötti kapcsolatot szavakkal adtuk meg. Szemléltesd ezt a kapcsolatot más módon!

Figure 3. A student's worksheet

based on the given rule. The students typically could hardly switch the representations after recognizing the covariant quantities (see the students' trial in Figure 3). By this, it can be concluded that in the cases of most participating students, the distinct representations of the function are built into their knowledge isolatedly from each other, the continuous development of the flexible ability of changing i.e. that of recognizing the connection between the representations when switching them wasn't emphasized enough after introducing the function concept.

After that the groups finished representing the covariant quantities in another form, they were asked to divide the given relationships into two categories based on that the same property should be true for the relationships of one category (i.e. we would have liked to have the representations of the would-be discovered concept). Considering the exercises, both groups emphasized the property of working with numbers (Exercises 1, 3, 4 and 5) and not working with numbers (Exercises 2 and 6). They prepared the category of the exercises being asked for based on this property. Though one of the groups asked whether *"Can only mathematical schemes be applied to the exercises?"*, we didn't give any induction for the students since our aim was to foster that the group could find the property needed for the categorisation on their own. By all of these we can conclude that when the function concept was introduced, the profound and thorough examination of the assignments of several types wasn't emphasized.

After that the teams explained the rules based on which they divide the exercises (*"in exercises 2 and 6 there are letters instead of numbers"*), we tried to give them a lead on what other qualities can be true for the finished exercise categories. We called their attention to the assignment i.e. that in exercises 1 and 2 we assigned only one "thing" (Exercise 1) and two "things" (Exercise 2) to a "thing". However, they didn't realize even at this point that it was the function concept which could be in the background. They noticed it only after that we had directed their attention to Exercise 4 where the function was represented by a formula. Then it became clear for them that some exercises represent functions and some other ones don't. So, by this it could be seen well that the students connect a function to a formula, to a rule of assignment given by a formula like the students in the studies of numbers (see Chapter *"Educational aspects of the development of the function concept"*). In other words, the students had a cognitive scheme[2] in connection with the function concept which considers the possibility of representation by an expression (by a formula) relevant information.

After analysing together the covariant quantities introduced in the exercises and phrasing together the (Dirichlet-Bourbaki) definition of the function (we note here that the students hardly managed phrasing it on their own), we clarified the domain and the range in the exercises, then we asked the groups to find examples and counterexamples for function connections from real life and name the sets of the domain and the range. There were interesting examples in the groups. *"No function– months assigned to seasons. Domain: seasons, range: months"; "Function–initial date assigned to wars. Domain: war, range: initial date".* The groups discussed the examples of other groups.

Investigation of relationships from engineering area

The aims of this part of the teaching experiment–using interactive activities– were: (1) to develop students' relation recognizing skills by the investigations of the assignments "number pair to number pair", "number to number pair"; (2) to show the difference between the representation of the assignments "number pair to number pair", "number to number pair" and the assignments "number to number" in the Cartesian coordinate system; (3) understanding relationship represented by a formula.

[2] `http://tanmester.tanarkepzo.hu/kognitiv_semak` (last download: 1.08.2018)

We presented different relationships ($\mathbb{R} \to \mathbb{R}^2$ functions) with which students often meet in everyday life and that have a wide application area in engineering:

(1) cycloidal (Figure 4) can be the graphic image for an $\mathbb{R} \to \mathbb{R}^2$ function which defines the connection between rotation and the location of a given point of a circle in the case of a circle rolling without slipping on a curve.

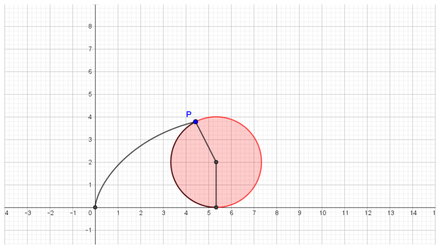


Figure 4. Simple cycloidal (own editing)

(2) throwing an object at an angle: in this case the motion path of the object graphic image of a vector-valued ($\mathbb{R} \to \mathbb{R}^2$) function which describes the relationship between the time and the position of the object (Figure 5) (Szíki et al., 2017).

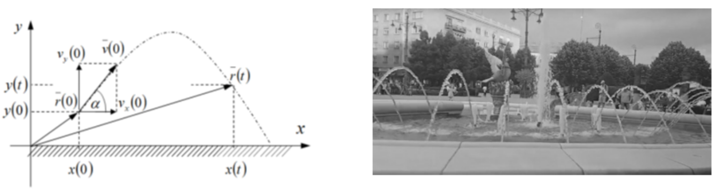


Figure 5. Throwing an object at an angle

Introductory task

Before these examinations of these relationships we considered it important that the students' meet assignments "number pair to number pair" ($\mathbb{R}^2 \to \mathbb{R}^2$ functions) in the frames of a simple exercise. For this end, we presented an axial reflection (as a congruent transformation) of a predefined point set by the help of a licence-free software called GeoGebra. This introduction exercise aimed at calling the students' attention to that the axes of the Cartesian coordinate system don't necessarily denote the values of the independent variable (x-axis) and the dependent variable (y-axis) in the case of functions. It can happen that we use the coordinate system to represent the range of the function–to represent the graphic image of the function.

Each group looked for the reflection of a point. The reflections of points appeared only if the children wrote appropriate coordinates into the white rectangle (Figure 6).

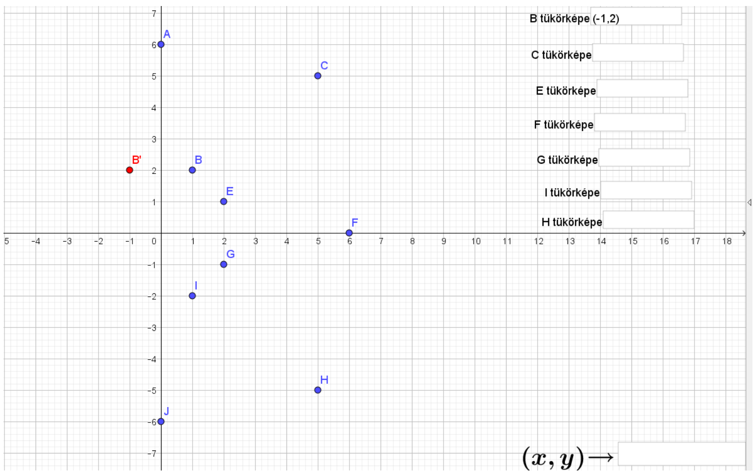


Figure 6. GeoGebra worksheet

Then we asked the students whether the assignment done (assign a point to a point) is a function? The students noticed that since *"...one point has only one reflection i.e. one point was assigned to each point so it is a function"*. We also mentioned that in this case, the domain and the range are sets of number pairs. Then we represent the relationship by a table, an arrow diagram, and we defined the rule of assignment in words and step-by-step by a formula ($(x, y) \rightarrow (-x, y)$) for the sake of developing the flexible ability of switching from a representation to another.

Investigation of the assignment "number to number pair"

- *Construction of the cycloidals using GeoGebra*

In order to have the students discover empirically the covariant quantities and the relationship between them, when constructing the cycloidal curve using GeoGebra (Figure 4) we asked the students to observe the process and give an answer to the following questions (which had been printed previously):

(1) was any kind of change occurred to the location of the point P?

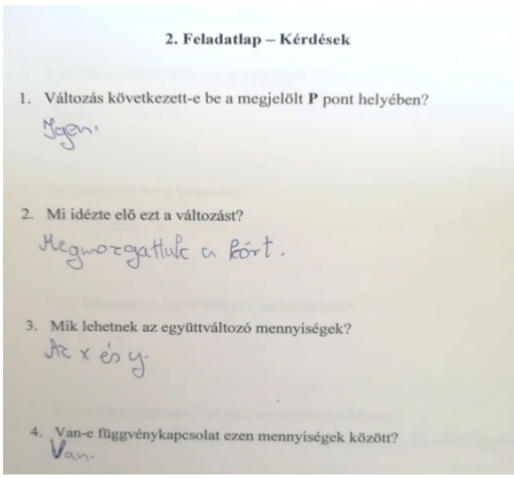
(2) what caused this change?

(3) which are the covariant quantities?

(4) is there a function connection between these quantities?

(5) if so, what is the independent and the dependent variable?

Of course, considering the chosen age group, it wasn't the aim that the students should represent the recognised relationship between the quantities by formulas. By constructing the graphic image of the function, the purpose was to recognize the covariant quantities in the formula defining the function i.e. to understand the substantial meaning of the function concept: a correspondence (assignment) between two sets that assigns every element of the first set to exactly one element of the second set. (In this example, to a rotation of a given degree we assign exactly one point of the plane.) We discussed together the written answers of the questions. In most cases, in questions 3, children answered using the usual notations: the covariant quantities are *"x and y"* (Figure 7[3]).



2. Feladatlap – Kérdések

1. Változás következett-e be a megjelölt **P** pont helyében?
Igen.

2. Mi idézte elő ezt a változást?
Megmozgattuk a kört.

3. Mik lehetnek az együttváltozó mennyiségek?
Az x és y.

4. Van-e függvénykapcsolat ezen mennyiségek között?
Van.

Figure 7. A student's worksheet

[3] ***Translation of the notes***: 1. Does anything change concerning the location of the point P? *"Yes."*
2. What caused this change? *"We moved the square."*
3. What can be the covariant quantities? *"x and y"*
4. Is there a function connection between them? *"Yes."*

Unfortunately, these usual notations can also take part in that children relate the function concept to the letters x and y and that its substantial meaning can thrust into the background. For this reason, during the development of the concept it is essential to examine several assignments and to denote variables in various ways in order to avoid that only one letter is related to the functions of the variables.

Then we emphasized again that these letters only denotes the quantities but it is necessary to give an explanation regarding that which quantities these notations were used for. So we tried to give an answer for question 3 together in such a way as to direct the students' attention to the answers of questions 1 and 2: the location of the point P was changed because of the rotation of the circle. At this point they already noticed that the location of the point P depends on the rotation i.e. *"the location of the point P depends on the measure of the rotation of the circle"*. In this case, we introduced the notation "t" for the rotation and the notation *(x,y)* for the location of the point, and then we wrote the assignment $(t \to (x, y))$ and defined the domain and the range.

After the discussion, we presented the application of cycloidals in engineering (cycloidal gears for wooden clocks[4]) which was very interesting for the children and they were glad to see this application area of functions.

- *Position-time relationship in the case of the throwing an object at an angle*

Before that we interpreted the representation of this type of functions given by a formula, we had examined the process of throwing the ball (like on Figure 1) by acting a situation in the classroom from the aspect of searching covariant quantities. The children didn't suggest anything for a long time, then we called their attention to the time elapsed:

Teacher: *" ... in your opinion, does anything change parallelly with the time in the course of throwing the ball?"*
Students: *"...it is the position of the ball"*

We clarified together that the time elapsed and the position of the ball can be covariant quantities: we assign the position of the ball to every moment. And then we constructed the orthogonal coordinate system by using Figure 1, denoting the positions of the ball in the course of the motion (by arrows only, without

[4] `http://www.wooden-clock.de/2012/02/19/free-open-source-cycloidal-gear-software/` (last download: Aug 1, 2018)

letters) by the points of the coordinate system. So we got a graphic image of a function which described the connection between the time elapsed and the position of the ball.

Teacher: *"So we have already clarified that a position of the ball can be denoted by a point in the orthogonal coordinate system. What do we need to know for depicting a point in the orthogonal coordinate system?"*

Students: *"The coordinates of the point."*

Students: *"So we need the x and y."*

We wrote down the usual notation of the coordinates of the point on the black board (x, y).

Teacher: *"Let us denote the time elapsed by the usual t. Since we have clarified that the position of the ball depends on this time and furthermore, the position can be denoted by a point having coordinates (x, y), do these coordinates also depend on t?"*

The students slightly hesitated. Then we directed their attention to Figure 1 again, we chose a point (a ball at a moment), we denoted the coordinates x and y of the point (the position of the ball) on the proper axes, as usual. Then we asked again:

Teacher: *"Do the coordinates of the point denoting the position of the ball depend on t?"*

Students: *"Yes, either x and y depend on it."*

At this point, using one type of denoting a function, we introduced the symbols denoting the coordinates' dependence on t: $(x(t), y(t))$. We described the relationship identified between the time elapsed and the position of the ball by the general formula $t \to (x(t), y(t))$, emphasizing that in this case, we don't assign a number to a number but a number to a number pair, and furthermore, this way we represented the range of our function in the coordinate system, the axes don't denote the values of the independent (x-axis) and the dependent (y-axis) variable but they show the range. (By an assignment "number to number" we explained the difference between the graphic representation of a "number to number" assignment and that of a "number to number pair" assignment.)

After the introductory exercise, by the following exercise we aimed at interpreting and variously representing functions similar to the presented one:

Exercise: *From the zero point of the orthogonal coordinate system we shoot a clay pigeon at an angle about* 64^o *at muzzle velocity* 15.65 *m/s (we ignore air resistance). At time t, the position of the clay pigeon is defined by the function* $t \to (7t, 14t - 5t^2)$.
1.) In the case of the given function, what are the independent and the dependent quantities?
2.) Give the positions of the clay pigeon at times $t = 0s, t = 0.5s, t = 1s$, $t = 1.5s, t = 2s$ *and* $t = 2.8s$.
3.) Give the connection between the time elapsed and the position of the clay pigeon by a table and an arrow diagram.
4.) Illustrate the motion of the clay pigeon by a graph.
5.) In how many seconds will the clay pigeon land?

Before solving the exercise, we performed the situation, interpreted the given rule, then we gave some time (5 minutes) for the students in order that they can try answering questions 1 and 2. Based on the previous exercise, they easily gave an answer for question 1 but we saw it needed to discuss the first part of question 2 (if $t = 0s$) collectively. After all this, they could identify the position at further times, and the representation by a table or an arrow diagram didn't mean a problem for them, neither.

We illustrated the motion path and answered question 5 together by the software GeoGebra, emphasizing that since the given function assigns a number pair to a number, when constructing the graphic image of our function we use the range of it in the coordinate system.

We animated the motion of the clay pigeon in GeoGebra (Figure 8) by showing the number pairs represented in the table created before. By presenting the motion with the help of an animation in GeoGebra, it didn't cause problem for the students to find the time of landing i.e. the moment when the point (the position of the clay pigeon) is on the x-axis i.e. the time t when the y-coordinate of the point equals 0.

Hereinafter, analogizing with the exercise presented, we showed such real life problems whose solution the application of the examined functions can help in. Since they are villager children, we presented watering by a garden hose as an example. We called their attention to the fact that we can say by the function describing the process for example that maximum how height can the water getting off from the hose at an angle about 64^o at muzzle velocity 15.65 m/s reach and in how many seconds will the clay pigeon land?

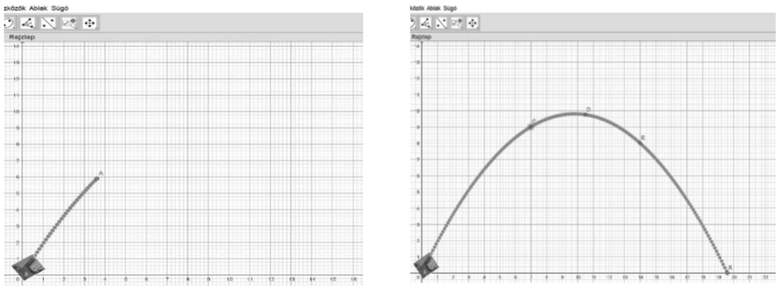


Figure 8. Representation of the motion of the clay pigeon by GeoGebra (own editing)

What is the function? – feedback of the teaching experiment

In order to explore what the students having participated in the teaching experiment understand about functions after that, 9 students gave written answers for some questions on the 28th of May, 2018: (1) what do we call as a function? (2) give an example of it; (3) give a counterexample of it. With all of these we wanted to know what influence had had the occupation on the development of the concept. In the Table 2 we summarized the students’ answers by the following aspects (the students were denoted by ordinal numbers):

(a) in the case of Question 1 we categorized the answers by the definition-categories of students stated in the study of (Vinner & Dreyfus, 1989):

- ”correspondence” – the function is a correspondence (assignment, connection) that assigns to every element of the first set exactly one element of the second set: *”An assignment is called a function if it assigns exactly one element of a set to an element of another set”* (student 1);
  ”A connection, assignment, coherence of quantities is called a function if exactly one element of a set is assigned to an element of another set” (student 5);
- into the category ”function initiative” we rated such answers in which there were only ”traces” considering the function concept: *”a coherence between two quantities is called a function”* (student 3);

(b) Which representations were used for answering Questions 2 and 3?

Ordinal number of students	**What is the function?**	**Example** representation	**Counter-example** representation
1	correspondence	verbal	verbal
2	incorrect answer (function initiative)	not satisfies the definition	not satisfies the definition
3	incorrect answer (function initiative)	arrow diagram	arrow diagram
4	relation of dependence	verbal	verbal
5	correspondence	verbal	verbal
6	correspondence	–	arrow diagram
7	correspondence	verbal	verbal
8	correspondence	arrow diagram	arrow diagram
9	incorrect answer (function initiative)	verbal	verbal

Table 2. Results of the post research of the "pilot study"

As it can be seen by the answers, most students who gave a definition of the "correspondence" category represented their example and counterexample verbally which could be built on the everyday experiences of the students. By this, we can conclude also that the students didn't connect the function concept to function connections (specific function) having studied at school and able to be defined by formulas but abstracting from this, they "projected" and explained verbally the function concept of the current stage of the concept development being in their imagination (Figure 9a[5]). In other words, concerning a function, it wasn't the property of representability which they considered relevant but the "one-to-one" quality. However, the 8th class students having participated in the former survey specified such examples of functions describable by formulas which they often meet in scholar Mathematics education (Figure 9b[6]) and they used

[5] ***Translation of the notes:*** b) Give an example of it.
"Lesson − > bell schedule
(a lesson has only one end)"
c) Give a counterexample of it.
"Family member − > relatives
(a family member can have several relatives)"
[6] b) Give an example of it. *"3-x"*

this concept image to decide whether a connection represented in a given form is a function connection or not (Szanyi, 2017a).

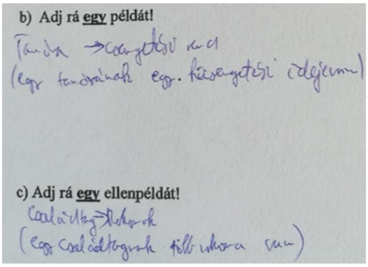


Figure 9a: Example and counterexample of a function (8th class, 2018)

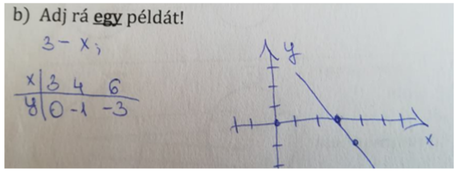


Figure 9b: Example of a function (8th class, 2016)

Conclusions

This study is an initial plan for a research which aims mainly to develop the function concept image of students. The teaching experiment presented contained activities built on each other by the principle of spirality, considering the age group and the mathematical knowledge of the chosen students. By this "pilot study", we wanted to know how the examinations of the relationships of engineering and everyday life can be built into the concept development process. Our experiences help basically plan the next research period.

By the experiences of the teaching experiment, we can conclude that the examination of the relationships found in everyday life not only raise the students' interest towards functions, but the activities presented were useful concerning the understanding of the substantial meaning of the concept. In contrast with the results of the surveys performed with 8th grade students and first-year university students (Szanyi, 2017b), as for the students having participated in the teaching experiment, it wasn't the representability of a function which they considered relevant but the "one-to-one" quality of it. Hence, our experiences confirmed the assumptions of the theoretical background i.e. it is expedient to develop the function concept with the help of real life situations, giving as much empirical material to the students as it is possible by studying "worlds" needed for the understanding.

The results strengthen the conclusions of (Vinner & Dreyfus, 1989) namely that the concept can't be learnt in one step but there is a long process leading here. But to evade any break during this process i.e. developing an incorrect concept image, we consider it reasonable to continuously follow the development process of the function concept in the elementary and secondary education in order to know the concept realization and the concept image of the students. This need could be seen clearly by the experiences gained in the first part of the class work and some previous research results (see the section ”Introduction”).

As we worked with a group few in numbers, we don't aim to generalize the results. The aim of the teaching experiment was to show the students functions from real life which are different from the specific functions leart at school, in order they can think on the function as a correpondence which is in everyday's life. Our experiences showed clearly that the improvement of educational practice being effective for the students can contribute to the concept development in large. In the teacher training necessary to demonstrate the importance of the applications of the function in real life using, for example, the results of the several teaching experiments, in order to give teachers strong focus on using the function to model real-life problems.

By all these, we consider building the following activities into the class work relevant:

- in spite of the fact that the function concept is already introduced in the 7th grade, we think that it is sufficient to redefine the concept in the 8th grade inductively, within the frames of discovery learning, playing and examining several assignments, in order that the activity can become personal by applying the various representations. So, in our opinion, in the first part of the class work it is expedient to play some assignments with the help of the children, aiming at storing the activity in the episodic remembrance (Csépe, 2007).
- in the course of the activities, it should be emphasized to deal with the concepts of the domain and the range, the (independent and the dependent) variables, with their notations and the direction of the assignment.
- at the end of the teaching experiment, we should examine its effectiveness and the know the students' thinking by a playing activity, with some exercises of concept identifying and concept realization.
- completing our experiment with problem situations that enlightens the function concept's role acting in problem solving.

The long-term effects of the teaching experiment presented above can be measured by a survey processed in the same student group based on which we can plan the next period (9th grade) of the concept development.

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