

Torsion of Truncated Hollow Spherical Elastic Body

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Abstract. This paper deals with the torsion of a body of rotation whose shape is a truncated hollow sphere. The material of the truncated hollow sphere is isotropic, homogeneous and linearly elastic. To solve the torsion problem, the theory of torsion of shafts of varying circular cross section is used, which is introduced by Michell and Föppl. Analytical solution is given for the shearing stresses and displacements. A numerical example illustrates the application of the presented solution. The results of the presented numerical example can be used as a benchmark problem to verify the accuracy of the results computed by finite element simulations.

Keywords: Torsion, body of rotation, truncated hollow sphere, elasticity

Introduction

The torsional deformation of a body of rotation is a very important chapter in the mechanics of structures [1, 2, 3, 4, 5, 6]. Paper by Bramble deals with a thick elastic spherical shell subjected to concentrated torques at the end-points of a diameter of the outer spherical boundary surface [7]. It is proven that the solution is uniquely determined and an explicit expression is derived for it. The author of [7] uses the fact that the stress function is closely related to an axially symmetric harmonic function defined in a seven-dimensional space. Pöschl studied the torsion problem of circular conical elastic body [8]. Book written by Arutyunyan and Abramyan [9] gives an analytical solution for the torsion of a hemisphere. For body of rotations, whose boundary surfaces are coordinate surfaces of an orthogonal curvilinear coordinate system, the closed form solutions are derived to the torsional boundary value problem [11]. There are several works on the problem of torsion of a body of rotation. It is not the purpose of this paper to provide a detailed list of the literature on this topic. The book of Arutyunyan and Abramyan [9], and the book of Lekhnitskii [12] present many works on the torsion of elastic shafts of varying circular cross section. In this paper the torsion of the 'hollow' truncated sphere is studied. The considered truncated sphere is made of homogeneous, isotropic and linearly elastic material. The formulation of the torsional boundary value problem is given in cylindrical coordinate system $Or\varphi z$ (Fig. 1). The meridian section of the truncated hollow sphere is shown in Fig. 2. The truncated hollow sphere occupies the space domain B in the three-dimensional space

$$B = \{(r, \varphi, z) | 0 \leq z \leq L, r_1(z) = \sqrt{R_1^2 - z^2} \leq r \leq r_2(z) = \sqrt{R_2^2 - z^2}, 0 \leq \varphi \leq 2\pi\}. \quad (1)$$

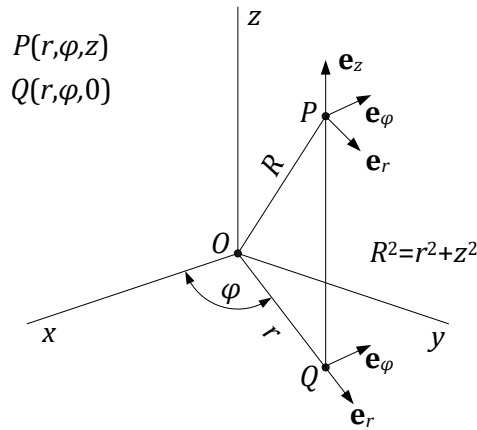


Figure 1. Cylindrical coordinates (r, φ, z)

The boundary curve of the meridian section A of body B

$$\partial A = \partial A_1 \cup \partial A_2 \cup \partial A_3 \cup \partial A_4 \tag{2}$$

as shown in Fig. 2. The boundary surface segments ∂B_1 and ∂B_2 are stress free. The tangential traction $\tau_{\varphi z}$ is not point-wisely prescribed on the end cross sections ∂B_3 and ∂B_4 , only the moments of tractions at the end cross sections about axis z are given. Denote T the applied torque on end cross sections ∂B_3 and ∂B_4 . ∂B_i is obtained from a complete rotation of ∂A_i ($i = 1, 2, 3, 4$) about axis z . According to the condition of mechanical equilibrium we have

$$T = \int_{\partial B_3} r \tau_{\varphi z}(r, \varphi, 0) da = \int_{\partial B_4} r \tau_{\varphi z}(r, \varphi, L) da, \tag{3}$$

where da denotes the area element on the boundary surface ∂B . Equation (3) formulates a weak traction boundary condition. This traction boundary value problem when the curved boundary part of the truncated hollow sphere is traction free and satisfies the weak boundary condition (3) can have solutions that differ a rigid body motion.

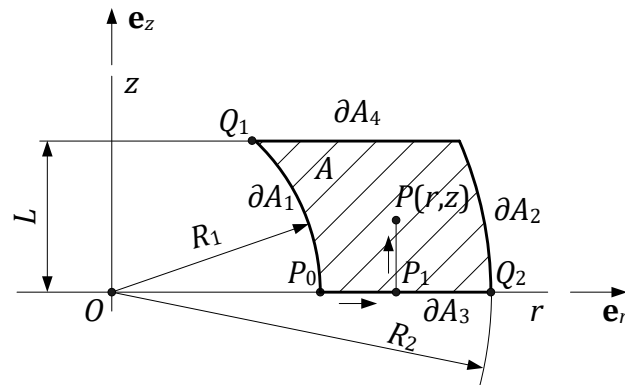


Figure 2. Meridian section of the truncated hollow sphere

1. Formulation of the problem

The Michell-Föppl's theory of the torsion of shafts of varying circular cross section is based on the next displacement field [2, 3, 4, 10]

$$\mathbf{u}(r, \varphi, z) = v(r, z)\mathbf{e}_\varphi(\varphi), \quad (r, \varphi, z) \in B. \quad (4)$$

The non-zero infinitesimal strains are

$$\gamma_{r\varphi} = \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{z\varphi} = \frac{\partial v}{\partial z}, \quad (5)$$

where $\gamma_{r\varphi}$ and $\gamma_{z\varphi}$ are shearing strains on the plane $\varphi = \text{constant}$. For isotropic homogeneous Hookean material the shearing stresses are given by

$$\tau_{r\varphi} = G\gamma_{r\varphi} = Gr \frac{\partial \psi}{\partial r}, \quad \tau_{z\varphi} = G\gamma_{z\varphi} = Gr \frac{\partial \psi}{\partial z}, \quad \psi = \frac{v(r)}{r}. \quad (6)$$

In the above equation G denotes the shear modulus of a homogeneous, isotropic and linearly elastic material. Since there are not body forces, the only non-trivial equation of equilibrium is

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial \tau_{z\varphi}}{\partial z} + \frac{2\tau_{r\varphi}}{r} = 0. \quad (7)$$

It is known that the general solution of equilibrium equation (7) in terms of first order stress function $U = U(r, z)$ can be represented as

$$r^2 \tau_{r\varphi} = -\frac{\partial U}{\partial z}, \quad r^2 \tau_{z\varphi} = \frac{\partial U}{\partial r}. \quad (8)$$

Combination of equation (6) with equation (8) gives the strain compatibility equation in terms of stress function $U = U(r, z)$

$$\frac{\partial^2 U}{\partial r^2} - \frac{3}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \quad (r, z) \in A. \quad (9)$$

The curved boundary surfaces ∂B_i ($i = 1, 2$) are traction free, that is

$$\tau_{r\varphi}(r, z)n_r + \tau_{z\varphi}(r, z)n_z = -\frac{n_r}{r^2} \frac{\partial U}{\partial z} + \frac{n_z}{r^2} \frac{\partial U}{\partial r} = -\frac{t_r}{r^2} \frac{\partial U}{\partial z} - \frac{t_z}{r^2} \frac{\partial U}{\partial r} = -\frac{\partial U}{\partial s} = 0. \quad (10)$$

In equation (10) n_r, n_z are the components of unit normal vector \mathbf{n} and the t_r, t_z are the components of unit tangential vector \mathbf{t} of the boundary curve $\partial A_3 \cup \partial A_4$ and s is arc-length defined on the boundary curve $\partial A_3 \cup \partial A_4$ (Fig. 3). Equation (10) shows that the stress function is constant on the curved boundary surface segments of the body of rotation. The torque T is taken across by the cross section at z is obtained as

$$T(z) = 2\pi \int_{r_1(z)}^{r_2(z)} r^2 \tau_{z\varphi} dr = 2\pi(U(r_2(z), z) - U(r_1(z), z)) = 2\pi U(r_2(z), z). \quad (11)$$

In equation (11) we prescribe that

$$U(r_1(z), z) = \text{constant} = 0, \quad 0 \leq z \leq L. \quad (12)$$

We can do it, since the $U(r, z)$ and $\bar{U}(r, z) = U(r, z) + K$ give the same stress and strain fields. The global condition of the mechanical equilibrium can be formulated as $T(z) = \text{constant}$.

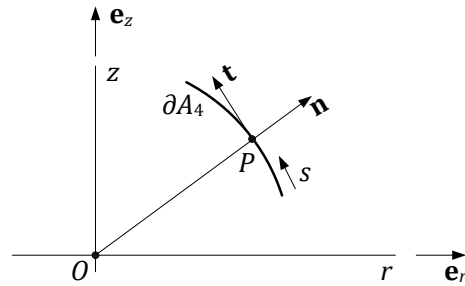


Figure 3. Illustration of tangential and normal unit vectors to the stress free boundary curve

2. Solution of the torsional problem

Our aim is to get the solution of the partial differential equation (9) under the boundary conditions

$$U(r_1(z), z) = 0, \quad U(r_2(z), z) = \frac{T}{2\pi}, \quad 0 < z < L. \quad (13)$$

We introduce a new variable t

$$t = r^2 + z^2, \quad (r, z) \in A \cup \partial A. \quad (14)$$

Simple computation shows that

$$\frac{d^2 U}{dt^2} \frac{\partial U}{\partial r} = 2r \frac{dU}{dt}, \quad \frac{\partial^2 U}{\partial r^2} = 4r^2 \frac{d^2 U}{dt^2} + 2 \frac{dU}{dt}, \quad (15)$$

$$\frac{\partial U}{\partial z} = 2z \frac{dU}{dt}, \quad \frac{\partial^2 U}{\partial z^2} = 4z^2 \frac{d^2 U}{dt^2} + 2 \frac{dU}{dt}. \quad (16)$$

Substitution of equations (15), (16) into the partial differential equation (9) gives an ordinary differential equation for $U = U(t)$

$$\frac{d^2 U}{dt^2} - \frac{1}{2t} \frac{dU}{dt} = 0. \quad (17)$$

The boundary condition (13)_{1,2} can be reformulated as

$$U(t_1) = 0, \quad t_1 = R_1^2, \quad U(t_2) = \frac{T}{2\pi}, \quad t_2 = R_2^2. \quad (18)$$

The solution of differential equation under the boundary condition (18)_{1,2} in terms of r and z is

$$U(r, z) = \frac{T}{2\pi} \frac{(\sqrt{r^2+z^2})^3 - R_1^3}{R_2^3 - R_1^3}, \quad (r, z) \in A \cup \partial A. \quad (19)$$

Application of formulae of shearing stresses $\tau_{r\varphi}$ and $\tau_{z\varphi}$ gives

$$\tau_{r\varphi}(r, z) = -\frac{3T}{2\pi} \frac{z \sqrt{r^2+z^2}}{r^2 (R_2^3 - R_1^3)}, \quad \tau_{z\varphi}(r, z) = \frac{T}{2\pi r} \frac{\sqrt{r^2+z^2}}{R_2^3 - R_1^3}. \quad (20)$$

Resulting shearing stress τ_φ can be computed as

$$\tau_\varphi(r, z) = \sqrt{\tau_{r\varphi}^2 + \tau_{z\varphi}^2} = \frac{T}{2\pi r^2} \frac{\sqrt{(r^2+z^2)(r^2+9z^2)}}{R_2^3 - R_1^3}. \quad (21)$$

Determination of the function $\psi = \psi(r, z)$ is based on equation (6). It is obvious

$$\frac{\partial \psi}{\partial r} = -\frac{3T}{2\pi G} \frac{z}{r^3} \frac{\sqrt{r^2+z^2}}{R_2^3-R_1^3}, \quad \frac{\partial \psi}{\partial z} = \frac{3T}{2\pi G r^2} \frac{\sqrt{r^2+z^2}}{R_2^3-R_1^3}. \quad (22)$$

Since we have

$$\frac{\partial^2 \psi}{\partial z \partial r} = -\frac{\partial}{\partial z} \frac{3T}{2\pi G} \frac{z}{r^3} \frac{\sqrt{r^2+z^2}}{R_2^3-R_1^3} = \frac{\partial^2 \psi}{\partial r \partial z} = \frac{\partial}{\partial r} \frac{3T}{2\pi G r^2} \frac{\sqrt{r^2+z^2}}{R_2^3-R_1^3}, \quad (23)$$

the formula (24) gives the explicit solution of the system of differential equations (Fig. 2)

$$\psi(r, z) = \int_{P_0 P_1} \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial z} dz + \int_{P_1 P} \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial z} dz. \quad (24)$$

A detailed computation gives

$$\psi(r, z) = \frac{3T}{4\pi G(R_2^3-R_1^3)} \left(\frac{z}{r} \sqrt{\left(\frac{z}{r}\right)^2 + 1} + \ln \left(\frac{z}{r} \sqrt{\left(\frac{z}{r}\right)^2 + 1} \right) \right). \quad (25)$$

The expression of the circumferential displacement $v = v(r, z)$ is as follows

$$v(r, z) = \frac{3Tr}{4\pi G(R_2^3-R_1^3)} \left(\frac{z}{r} \sqrt{\left(\frac{z}{r}\right)^2 + 1} + \ln \left(\frac{z}{r} \sqrt{\left(\frac{z}{r}\right)^2 + 1} \right) \right). \quad (26)$$

It is evident the cross section at $z = 0$ is fixed, that is

$$\psi(r, 0) = 0, \quad v(r, 0) = 0. \quad (27)$$

From equations (6) and (8) it follows that

$$\frac{\partial U}{\partial z} = -Gr^3 \frac{\partial \psi}{\partial r}, \quad \frac{\partial U}{\partial r} = Gr^3 \frac{\partial \psi}{\partial z}. \quad (28)$$

Since

$$\frac{\partial^2 U}{\partial r \partial z} = -Gr^3 \frac{\partial^2 \psi}{\partial r^2} = \frac{\partial^2 U}{\partial z \partial r} = Gr^3 \frac{\partial^2 \psi}{\partial z^2}, \quad (r, z) \in A \cup \partial A \quad (29)$$

the function $\psi = \psi(r, z)$ satisfies the partial differential equation written below, in the domain A

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad (r, z) \in A. \quad (30)$$

The stress free boundary condition on the curved boundary is expressed as

$$r \frac{\partial \psi}{\partial r} + z \frac{\partial \psi}{\partial z} = 0, \quad (r, z) \in \partial A_1 \cup \partial A_2. \quad (31)$$

Here, it is used

$$n_r = \frac{r}{R_i}, \quad n_z = \frac{z}{R_i}, \quad (r, z) \in \partial A_i, \quad (i = 1, 2), \quad (32)$$

$$\tau_{r\varphi} = Gr^3 \frac{\partial \psi}{\partial r}, \quad \tau_{z\varphi} = Gr^3 \frac{\partial \psi}{\partial z}. \quad (33)$$

3. Numerical example

In the numerical example the following data are used $R_1 = 0.036$ m, $R_2 = 0.075$ m, $L = 0.03$ m, $G = 1.6 \times 10^8$ Pa, $T = 10$ kNm. The contour lines of the stress function $U = U(r, z)$ and the function $\psi = \psi(r, z)$ are shown in Fig. 4 and Fig. 5. The resulting shearing stress reaches its maximum value at point

Q_1 of the cross section and the minimum value of the resulting shearing stress is at point Q_2 of the cross section (Fig. 2). From formula (21) we get

$$\tau_\varphi(Q_1) = 35.5427 \text{ MPa}, \quad \tau_\varphi(Q_2) = 4.2416 \text{ MPa}. \quad (34)$$

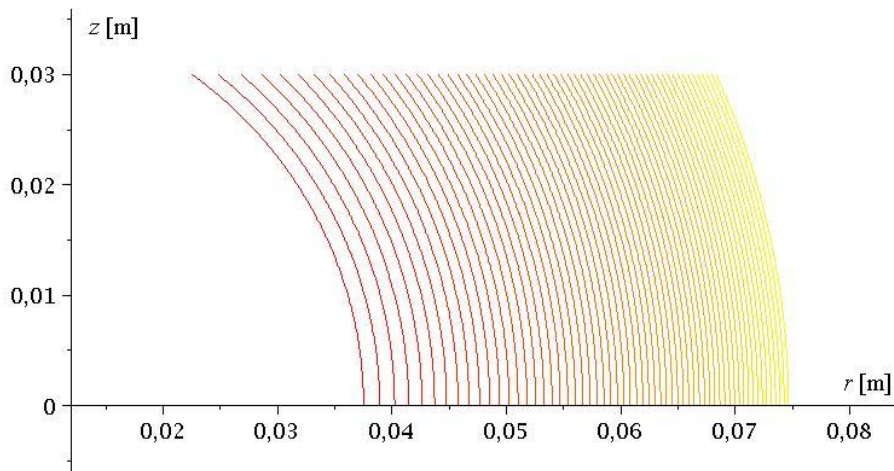


Figure 4. Contour lines of stress function $U = U(r, z)$

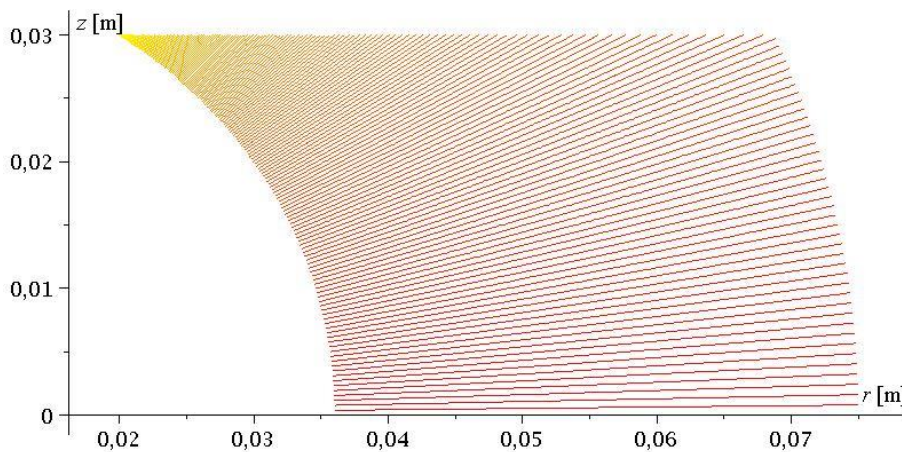


Figure 5. Contour lines of function $\psi = \psi(r, z)$

Conclusions

An analytical solution is presented for the torsion problem of truncated hollow spherical body whose material is isotropic, homogeneous and linearly elastic. It is assumed that the deformations are small and the formulation of linearized theory of elasticity can be used. Numerical example illustrates the applications of the derived formulae.

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