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Pure Bending of Homogenous Isotropic Elastic Curved Beam

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Abstract. In this paper a detailed analysis is given for the pure bending problem of curved beams. The material of the curved beam is homogenous isotropic linearly elastic. The mantle of the curved beam is stress free and there is no body force on the curved beam. The plane of the curvature of the beam is the plane of symmetry for the whole beam. Paper gives the expressions of circumferential and radial normal stresses. A strength of material approach is used to derive the governing equations. A numerical example illustrates the application of the presented solutions.

Keywords: Bending, Curved Beam, Linearly Elastic, Strength of Materials Solution

Introduction

Figure 1 shows the considered curved beam and its cross section.

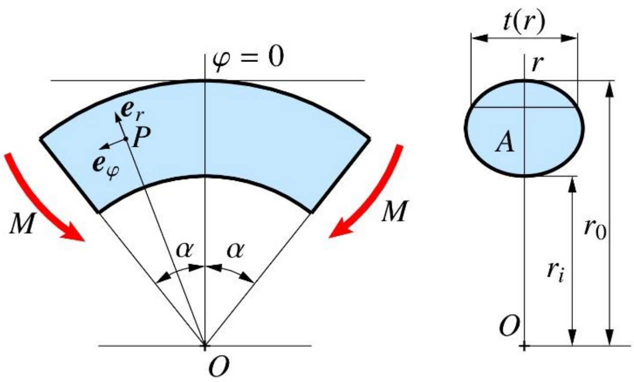

Figure 1. Curved beam loaded by bending moment.

The curved beam occupies the space domain B

$$B = \left\{ (r,\varphi,z) |\, r_i \le r \le r_0, -\frac{t(r)}{2} \le z \le \frac{t(r)}{2}, -\alpha \le \varphi \le \alpha \right\}. \tag{1}$$

The cross section of the beam is denoted by A and the boundary curve of A is ∂A. It is evident that

$$\partial B = \{(r,z) \in \partial A, -\alpha \le \varphi \le \alpha\}. \tag{2}$$

It is assumed that the cross section is symmetric, and it remains plane and just rotates about the neutral axis as in the case of straight beam. The significant stress component is the hoop stress σ_φ. Several

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textbooks deal with the elastic curved beam, it is a usual chapter the mechanics of curved beam in books of mechanics of solids [1,2,3,4]. In this paper a detailed analysis is given which deals with the determination of the radial stresses too.

1. Governing equations for stresses

The formulation of the governing equations are given in cylindrical coordinate system $Or\varphi z$. The unit vectors of the cylindrical coordinate system are $\boldsymbol{e}_r, \boldsymbol{e}_\varphi, \boldsymbol{e}_z$ (see Figure 1). The solution is based on the following displacement field [6]

$$\boldsymbol{u}(r,\varphi,z) = U(\varphi)\boldsymbol{e}_r + \left(r\phi(\varphi) + \frac{dU}{d\varphi}\right)\boldsymbol{e}_\varphi, \tag{3}$$

where $\boldsymbol{u}$ is the displacement vector, $U = U(\varphi)$ is the radial displacement, $\phi = \phi(\varphi)$ is the cross-sectional rotation. Application of the strain-displacement relationships we obtain [1,2]

$$\varepsilon_r = \varepsilon_z = \gamma_{r\varphi} = \gamma_{z\varphi} = 0, \tag{4}$$

$$\varepsilon_\varphi = \frac{1}{r}\left(\frac{d^2U}{d\varphi^2} + U\right) + \frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\,. \tag{5}$$

Here $\varepsilon_r, \varepsilon_\varphi$ and ε_z are the normal strains; $\gamma_{r\varphi}, \gamma_{z\varphi}$ and γ_{rz} are the shearing strains. Hooke's law gives the result for the hoop stress σ_φ

$$\sigma_\varphi = E\left(\frac{W}{r} + \vartheta\right). \tag{6}$$

where

$$W = \left(\frac{d^2U}{d\varphi^2} + U\right), \qquad \vartheta = \frac{\mathrm{d}\phi}{\mathrm{d}\varphi}. \tag{7}$$

It is evident in the case of pure bending W and ϑ do not depend on the polar angle φ. The position of the neutral axis is determined by the radius R. From formula (6) we obtain

$$W = -R\vartheta, \tag{8}$$

and

$$\sigma_\varphi = E\left(1 - \frac{R}{r}\right). \tag{9}$$

The normal stress resultant N is computed as

$$N = \int_A \sigma_\varphi \,\mathrm{d}A = E\left[A - R\int_{\boldsymbol{A}} \frac{\mathrm{d}A}{r}\right]\vartheta. \tag{10}$$

Since $N = 0$, we have

$$R = \frac{A}{\int \frac{\mathrm{d}A}{r}} = \frac{\int_{r_i}^{r_0} t(r)\mathrm{d}r}{\int_{r_i}^{r_0} \frac{t(r)}{r}\mathrm{d}r}. \tag{11}$$

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The bending moment in terms of ϑ is as follows

$$M = \int_A r\sigma_\varphi \mathrm{d}A = E\left(\int_{r_i}^{r_0} r\, t(r)\, \mathrm{d}r - R\int_{r_i}^{r_0} t(r)\, \mathrm{d}r\right)\vartheta = EA(r_c - R)\vartheta = EAe\vartheta, \tag{12}$$

that is

$$\vartheta = \frac{M}{EAe}, \quad e = r_c - R, \qquad r_c = \frac{1\int_A r\, \mathrm{d}A}{A}. \tag{13}$$

From the Cauchy inequality relation

$$\int_A \left(\frac{1}{\sqrt{r}}\right)^2 \mathrm{d}A \int_A \left(\sqrt{r}\right)^2 \mathrm{d}A \geq \left(\int_A \frac{1}{\sqrt{r}}\sqrt{r}\, \mathrm{d}A\right)^2 \tag{14}$$

it follows that

$$R \leq r_c. \tag{15}$$

Combination of equation (9) with equation (13) gives

$$\sigma_\varphi = \frac{M}{Ae}\left(1 - \frac{R}{r}\right). \tag{16}$$

The hoop stress resultant N_φ is defined as

$$N_\varphi(r) = t(r)\sigma_\varphi(r) = \frac{M\, t(r)}{A\, e}\left(1 - \frac{R}{r}\right). \tag{17}$$

According to the equilibrium equation

$$\frac{\mathrm{d}}{\mathrm{d}r}[rN_r(r)] = N_\varphi(r), \tag{18}$$

where N_r is defined as

$$N_r(r) = t(r)\sigma(r), \tag{19}$$

here $\sigma_r = \sigma_r(r)$ is the radial normal stress. The solution of the differential equation (18) under the initial condition

$$N_r(r_i) = 0 \tag{20}$$

is as follows

$$N_r(r) = \frac{1}{r}\int_{r_i}^{r} N_\varphi(\rho)\, \mathrm{d}\rho. \tag{21}$$

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Detailed expression of the radial stress resultant N_r is

$$N_r(r) = \frac{M}{A\,e\,r}\int_{r_i}^{r} t(\rho)\left(1-\frac{R}{\rho}\right)\,\mathrm{d}\rho. \tag{22}$$

$N_r = N_r(r)$ satisfies the following boundary condition

$$N_r(r_0) = 0. \tag{23}$$

The validity of this equation is follows from equation (24)

$$N_r(r_0) = \frac{M}{A\,e\,r}\int_{r_i}^{r} t(\rho)\left(1-\frac{R}{\rho}\right)\mathrm{d}\rho = \frac{N}{r_0} = 0. \tag{24}$$

From equations (19) and (22) we obtain

$$\sigma_r(r) = \frac{M}{A\,E\,r\,t(r)}\int_{r_i}^{r} t(\rho)\left(1-\frac{R}{\rho}\right)\mathrm{d}\rho. \tag{25}$$

The formula of the von Mises stresses is represented by equation (26)

$$\sigma(r) = \sqrt{\sigma_r^2(r) - \sigma_r(r)\sigma_\varphi(r) + \sigma_\varphi^2(r)}\,. \tag{26}$$

2. Governing equations for deformation

The expression of the displacement vector $\boldsymbol{u}$ can be represented as

$$\boldsymbol{u}(r,\varphi,z) = U(\varphi)\boldsymbol{e}_r + [r\phi(r) + V(\varphi)]\boldsymbol{e}_\varphi. \tag{27}$$

where

$$V(\varphi) = \frac{\mathrm{d}U}{\mathrm{d}\varphi}. \tag{28}$$

The following displacement *"boundary conditions"* are applied

$$U(0) = 0, \quad V(0) = 0, \quad \phi(0) = 0. \tag{29}$$

We starting from equation (30)

$$U'' + U = W = -\frac{M\,R}{E\,A\,e}. \tag{30}$$

From this equation it follows that

$$U(\varphi) = -\frac{M\,R}{E\,A\,e} + c_1\cos\varphi + c_2\sin\varphi \tag{31}$$

and

$$V(\varphi) = \frac{\mathrm{d}U}{\mathrm{d}\varphi} = -c_1\sin\varphi + c_2\cos\varphi. \tag{32}$$

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Combination of equation (29) with equations (31) and (32) gives

$$c_1 = \frac{M\ R}{E\ A\ e}, \qquad c_2 = 0. \tag{33}$$

Substitution equation (33) into equation (29) gives

$$U(\varphi) = \frac{M\ R}{E\ A\ e}(\cos\varphi - 1), \tag{34}$$

$$V(\varphi) = -\frac{M\ R}{E\ A\ e}\sin\varphi. \tag{35}$$

The cross sectional rotation $\phi = \phi(\varphi)$ is obtained from equation (13)

$$\phi(\varphi) = \frac{M}{E\ A\ e}\varphi. \tag{36}$$

3. Numerical Example

In the numerical example we consider an elliptical cross section (see Figure 2).

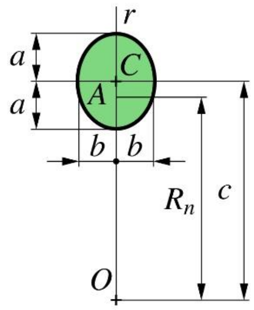

Figure 2. Elliptical cross section.

The equation of the boundary contour is

$$\frac{(r-c)^2}{a^2} + \frac{z^2}{b^2} = 1. \tag{37}$$

We introduce the elliptical coordinate

$$r = c + a\lambda\cos\vartheta\,, \qquad z = a\lambda\sin\vartheta\,. \tag{38}$$

The expression of the area element dA is

$$\mathrm{d}A = \lambda ab \tag{39}$$

$$R = \frac{A}{\int_A \frac{\mathrm{d}A}{r}} = \frac{a\ b\ \pi}{\int_{\lambda=0}^{1}\int_{\vartheta=0}^{2\pi}\frac{\lambda\ a\ b}{c + a\ \lambda\ cos\vartheta}\mathrm{d}\lambda\ \mathrm{d}\vartheta} = \frac{\pi}{\int_{\lambda=0}^{1}\int_{\vartheta=0}^{2\pi}\frac{\mathrm{d}\lambda\ \mathrm{d}\vartheta}{c + a\lambda\cos\vartheta}}. \tag{40}$$

It is evident that

$$e = c - R. \tag{41}$$

A closed form formula is given for R in book by Pissaraiko et al. [7]

$$R = \frac{a^2}{2(c - \sqrt{c^2 - a^2})}. \tag{42}$$

The following numerical data are used

$$a = 0.015 \text{ m}, \quad b = 0.06 \text{ m}, \quad c = 0.04 \text{ m}, \quad E = 10 \times 10^{11} \text{ Pa}, \quad M = 104 \text{ Nm}. \tag{43}$$

From formula (42) we obtain

$$R = 0.03854049623 \text{ m}. \tag{44}$$

The plot of $\sigma_\varphi = \sigma_\varphi(r)$ is shown in Figure 3.

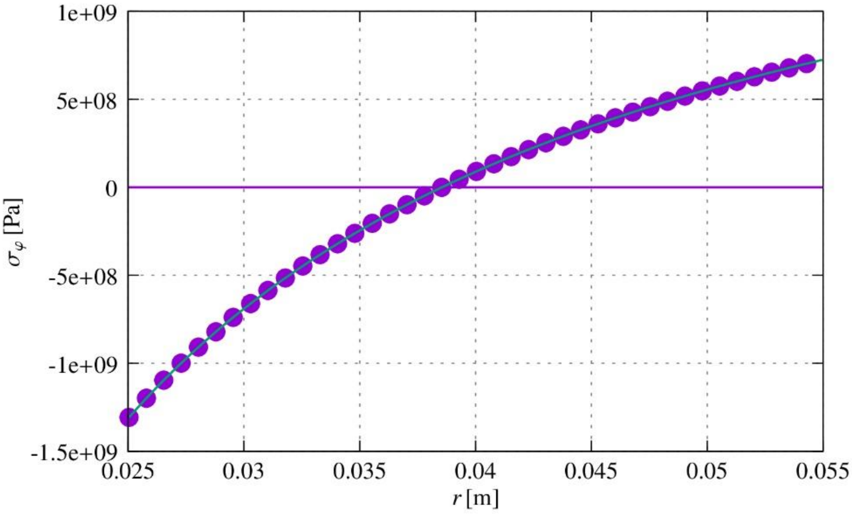

Figure 3. The plot of the hoop stress σ_φ as a function of r.

The graph of $N_\varphi = N_\varphi(r)$ is given in Figure 4.

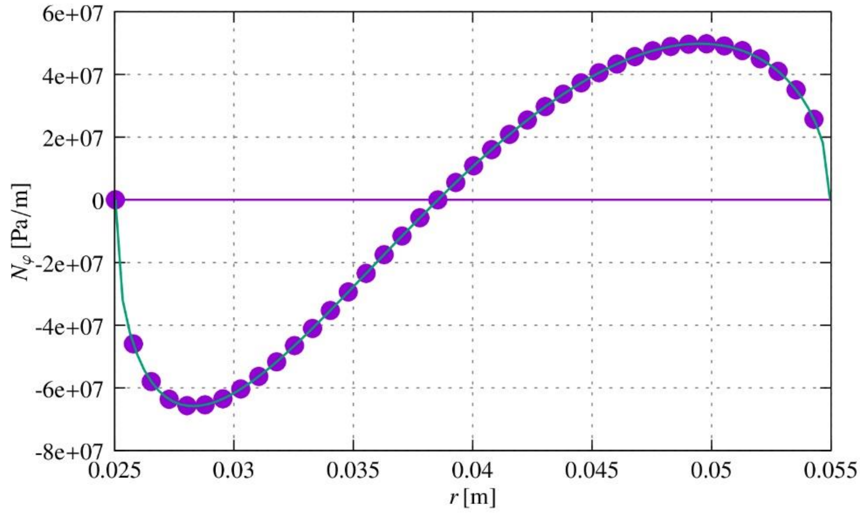

Figure 4. The plot of N_φ as a function of r.

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In Figure 5 the graph of the normal stress resultant N_r is shown.

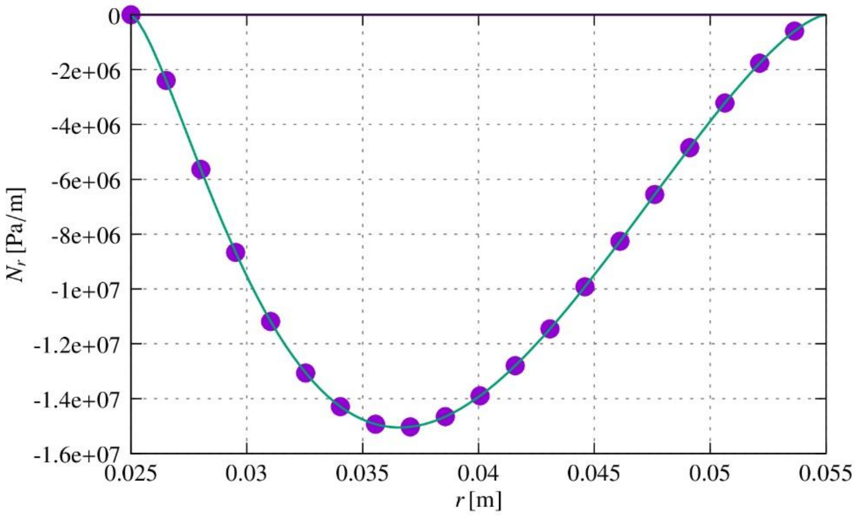

Figure 5. The graph of N_r as a function of r.

The graph of the radial normal stress is represented in Figure 6.

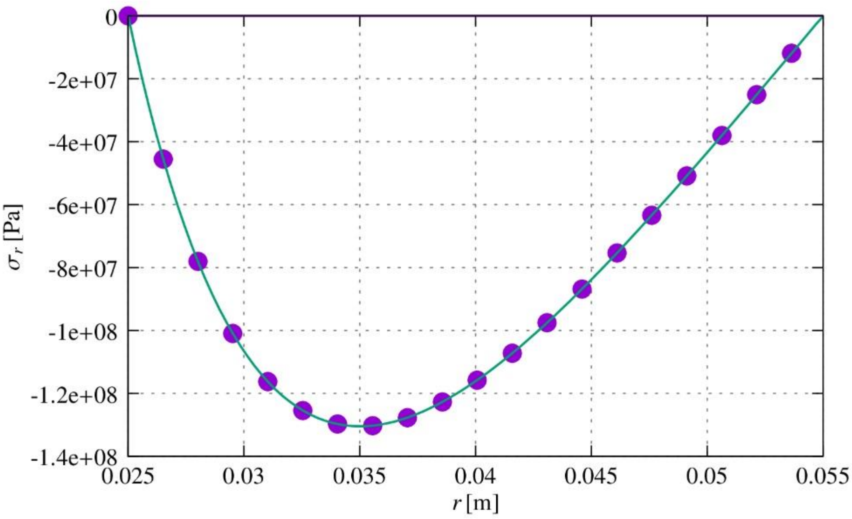

Figure 6. The radial normal stress as a function of r.

The von Mises stress is given in Figure 7 and we have

$$\sigma_0 = \max \sigma(r) = 1.3125 \times 10^8 \text{ Pa} \qquad r_i \le r \le r_0 \tag{45}$$

For $\varphi_1 = -\frac{\pi}{8}, \varphi_2 = \frac{\pi}{8}$ the displacements $U = U(\varphi), V = V(\varphi)$ and cross sectional rotation are shown in Figures 7, 8 and 9.

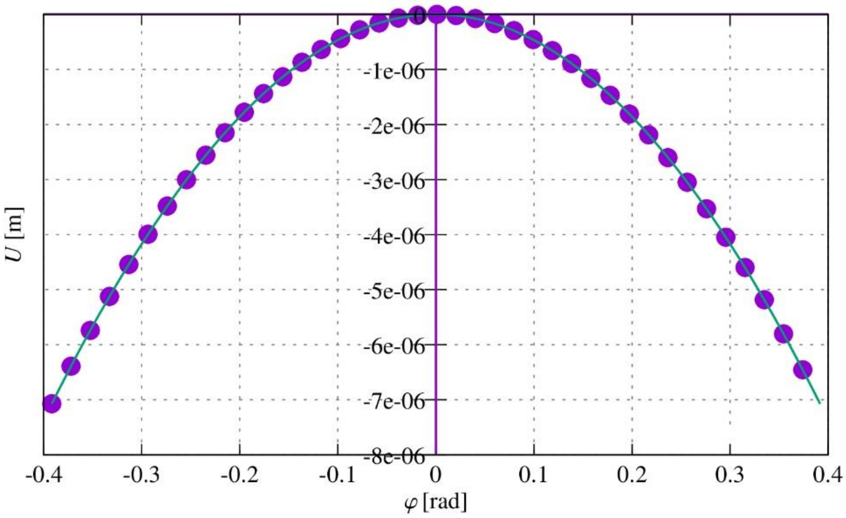

Figure 7. The plot of the radial displacement.

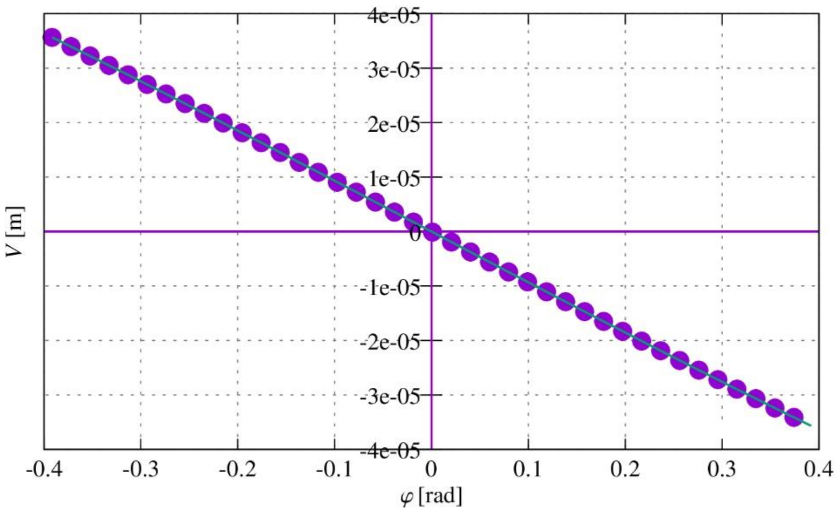

Figure 8. The plot of the circumferential displacement component $V = V(\varphi)$.

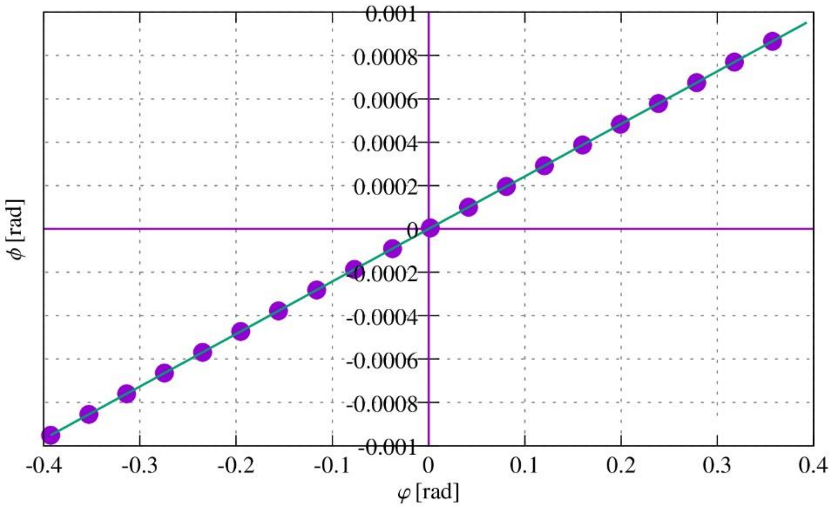

Figure 9. The plot of the cross-sectional rotation.

Conclusions

Paper deals with the pure bending problem of curved beam. The material of the beam is linearly elastic, isotropic and homogenous. In-plane deformation of the curved beam is considered. Paper gives the expressions of the radial and hoop normal stresses. A strength of material solution is presented. The numerical results of this study can be used as benchmark solution to check the results of usual numerical methods such as finite elements, finite differences, etc.

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