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Teaching the Analysis of Newton's Cooling Model to Engineering Students

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Abstract. To apply mathematical methods to physical or other real life problem, we have to formulate the problem in mathematical terms. It means that, we have to construct the mathematical model for the problem. Many physical problems shows the relationships between changing quantities. The rates of change are represented mathematically by derivatives. In this case the mathematical models involve equations relating an unknown function and one or more of its derivatives. These equations are the differential equations. In this article, teaching the analysis of Newton's cooling model to engineering students is presented as one of the applications of separable differential equations.

Keywords: differential equations, Newtons's cooling model, applied analysis

Introduction

Much of differential equations is devoted to learning mathematical techniques that are applied in later courses in other sciences.

In this article we mention a few such applications. The mathematical model generally simplier than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved.

A good mathematical model has two important properties:

- It is sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy.

We will give some examples for mathematical models involving separable differential equations related to Newton's Law of Cooling.

All the examples in this article deal with functions of time, which we denote by t . If x is a differentiable function of t, x' denotes the derivative of x with respect to t, thus $x' = \frac{dx}{dt}$.

In many cases, the preparation of the mathematical model for students is a serious problem.

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Describing the model when the temperature is constant

According to *Newton's law of cooling* the rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings. The temperature of the surroundings is sometimes called the ambient temperature. If $T(t)$ is the temperature at time t of the object and T_a is the temperature of its surroundings, then

$$T'(t) = -k \cdot (T(t) - T_a),$$

where k is a positive constant. This differential equation is separable.

This mathematical model it can be used well when studying a small object in a large, fixed temperature, environment.

First, we thinking a little about the sign of the constant of proportionality. At any time t there are three cases.

- If $T(t) > T_a$, that is, if the body is warmer than its surroundings, we would expect heat to flow from the body into its surroundings and so we would expect the body to cool off so that $T'(t) < 0$, thus $k > 0$.
- If $T(t) < T_a$, that is, if the body is cooler than its surroundings, we would expect heat to flow from the surroundings into the body and so we would expect the body to warm up so that $T'(t) > 0$, thus $k > 0$.
- If $T(t) = T_a$, that is the body and its environment have the same temperature, we would not expect any heat to flow between the two and so we would expect that $T'(t) = 0$. This does not impose any condition on k.

To the solution of the differential equation let denote $g(t) = -k$ and $h(T) = T - T_a$. In the general theory of the seprable differential equation, we get that

$$\int g(t)\ dt = \int \frac{1}{h(T)}\ dT \quad \Rightarrow \quad \int -k\ dt = \int \frac{1}{T - T_a}\ dT.$$

If we calculate the integrals, then the

$$-k \cdot t = \ln|T - T_a| + c$$

equality holds. If we solve this equation to variable T then we get that

$$T(t) = C \cdot e^{-k \cdot t} + T_a.$$

Let T_0 is the temperature of the body when $t = 0$, that is $T_0 = T(0)$. With this notation $T_0 = C + T_a$, thus $C = T_0 - T_a$. The general solution of the model

$$T(t) = (T_0 - T_a) \cdot e^{-k \cdot t} + T_a.$$

We note that

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} (T_0 - T_a) \cdot e^{-k \cdot t} + T_a = T_a,$$

which independent of T_0. In this model the air temperature is constant.

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Some concrete situation, when we can use the general model

- Forensic Science
- Cooling Down of Hot beverages
- Measuring Temperature of a Heated Metal
- Melting Ice-Cream

Concrete example 1

We would like to know the time at which a person died. In particular, we know the investigator arrived on the scene at 10:23 pm, which we will call t hours after death. At 10:23, the temperature of the body was found to be $26.67\,^0C$. One hour later, the body was found to be 25.83. Our known data that, the temperature of environment, $T_a = 20^0C$ and $T_0 = 37^0C$. Substitute the data to the equation

$$T(t) = (T_0 - T_a) \cdot e^{-k \cdot t} + T_a,$$

we get that $T(t) = 17 \cdot e^{-k \cdot t} + 20$, where t corresponds to $10:23$ and represents the time (in hours) since death. Since $T(t+1) = 17 \cdot e^{-k \cdot (t+1)} + 20$, therefore

$$26.67 = 17 \cdot e^{-k \cdot t} + 20 \text{ and}$$

$$25.83 = 17 \cdot e^{-k \cdot (t+1)} + 20.$$

From the first equation we get that $e^{-k \cdot t} = \frac{6.67}{17}$, hence $25.83 = 17 \cdot \frac{6.67}{17} \cdot e^{-k} + 20$. Solving this equation the value of k will be $k = -\ln\left(\frac{5.83}{6.67}\right) \approx 0.1346$.

Now that we have a value for k, we can use this to solve for the remaining unknown, t. We have to find the value of t, when $T(t) = 26.67$, that is we have to solve the equation

$$26.67 \approx 17 \cdot e^{-0.1346 \cdot t} + 20 \quad \Rightarrow \quad t \approx 7.$$

The final result means that the detective arrived on scene at 10:23 pm (t hours after death), the individual must have died 7 hours prior to 10:23 pm that is at approximately 3:23 pm.

Concrete example 2

Suppose that a cup of coffee is initially at a temperature of 95^0C and is placed in a 25^0C room. After a minute the temperature will be 80^0C.

Newton's law of cooling says that

$$T'(t) = -k \cdot (T(t) - 25).$$

The solution of the differential equation is

$$T(t) = 70 \cdot e^{-k \cdot t} + 25.$$

It is known that $80 = 70 \cdot e^{-k} + 25$, that is $k = -\ln\frac{11}{14}$. We get that, the time-temperature function is

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$$T(t) = 70 \cdot e^{\ln\frac{11}{14}\cdot t} + 25 = 70 \cdot \left(\frac{11}{14}\right)^{t} + 25.$$

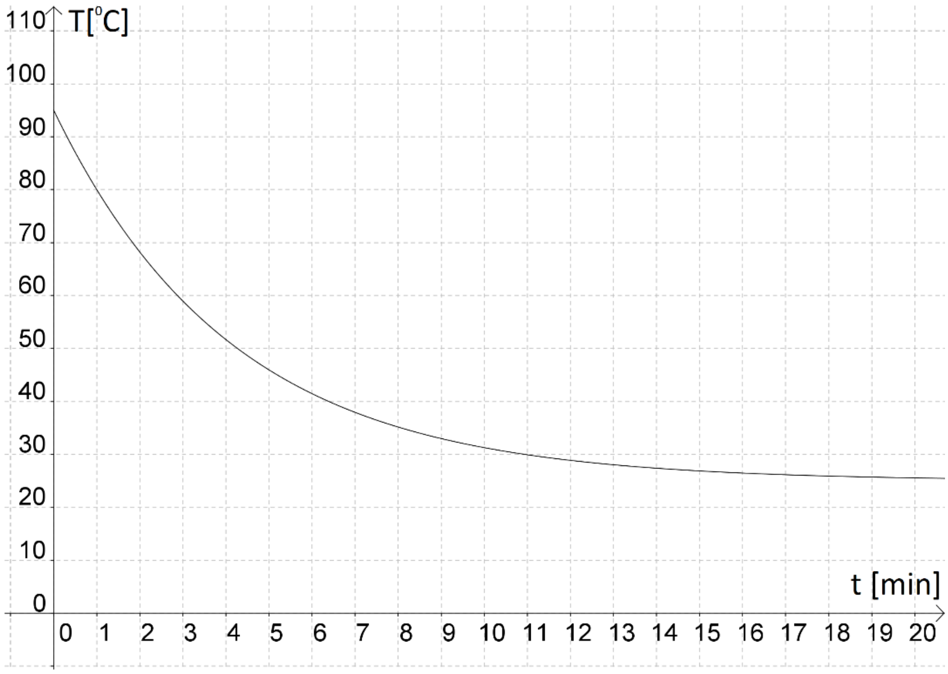

Figure 1. Temperature as a function of time

A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is 20^0C. After one minute the water has temperature 22^0C and after two minutes it has temperature 23^0C. What is the outdoor temperature?

To the solution, we have to use the formula

$$T(t) = (T_0 - T_a) \cdot e^{-k\cdot t} + T_a$$

again. We leave the solution of this exercise to the reader.

Description of the model for variable environmental temperatures

In the model described above, the temperature of the environment was constant. However, this condition is not always applicable.

Consider, for example, the simple situation that it does not matter if a cup of coffee cools down in a room or a huge cauldron of molten metal cools down in a room. The difference between the two situations is that the heat lost by the coffee probably will not raise the temperature of the room appreciably, but the heat loss from the cooling metal does. In this second situation, we have to use a model that takes into account the heat exchange between the object and the medium.

In this situation we have the equation

$$T'(t) = -k \cdot T(t) + k \cdot T_a(t).$$

From this equation we get that $T'(t) + k \cdot T(t) = k \cdot T_a(t)$. Multiplying both sides $e^{k\cdot t}$,

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$$T'(t) \cdot e^{k \cdot t} + k \cdot T(t) \cdot e^{k \cdot t} = k \cdot e^{k \cdot t} \cdot T_a(t),$$

that is $\left(T(t) \cdot e^{k \cdot t}\right)' = k \cdot e^{k \cdot t} \cdot T_a(t)$. Integrating both sides it follows that

$$T(t) \cdot e^{k \cdot t} - T(t_0) \cdot e^{k \cdot t_0} = k \cdot \int_{t_0}^{t} e^{k \cdot \tau} \cdot T_a(\tau)\, \mathrm{d}\tau.$$

Concrete example

Let us model outdoor temperature as a cosine with minimum at midnight:

$$T_a(t) = 20 - 10 \cdot \cos\left(\frac{2\pi \cdot t}{24}\right),$$

where t is time in hours, and $T_a(t)$ is the temperature in Celsius degrees. Assume at a party someone forgets a beer at pre-party $t = 22$ (that is 22:00) in evening but finds it again at after-party $t = 26$ (that is 2:00). How can we approach this problem of calculating how much warmer the beer has gotten?

To the solution in fist step we have to integrate the function

$$e^{k \cdot \tau} \cdot \left(20 - 10 \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right)\right).$$

Using formula for partial integration, we get that

$$\int e^{k \cdot \tau} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) \mathrm{d}\tau = \frac{e^{k \cdot \tau}}{k} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) + \int \frac{e^{k \cdot \tau}}{k} \cdot \sin\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi}{12}\, \mathrm{d}\tau.$$

Integral by parts again, we get that

$$\int e^{k \cdot \tau} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) \mathrm{d}\tau = \frac{e^{k \cdot \tau}}{k} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) + \frac{e^{k \cdot \tau}}{k^2} \cdot \sin\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi}{12} - \int \frac{e^{k \cdot \tau}}{k^2} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi^2}{144}.$$

From the last equation it follows that

$$\int \left(1 + \frac{\pi^2}{144k^2}\right) \cdot e^{k \cdot \tau} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) \mathrm{d}\tau = \frac{e^{k \cdot \tau}}{k} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) + \frac{e^{k \cdot \tau}}{k^2} \cdot \sin\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi}{12}.$$

This equation gives that

$$\int e^{k \cdot \tau} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) \mathrm{d}\tau = \left(\frac{e^{k \cdot \tau}}{k} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) + \frac{e^{k \cdot \tau}}{k^2} \cdot \sin\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi}{12}\right) \cdot \frac{144k^2}{144k^2 + \pi^2}.$$

Using this result

$$\int e^{k \cdot \tau} \cdot \left(20 - 10 \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right)\right) \mathrm{d}\tau$$
$$= \frac{20e^{k \cdot \tau}}{k} - 10 \cdot \left(\frac{e^{k \cdot \tau}}{k} \cdot \cos\left(\frac{2\pi \cdot \tau}{24}\right) + \frac{e^{k \cdot \tau}}{k^2} \cdot \sin\left(\frac{2\pi \cdot \tau}{24}\right) \cdot \frac{\pi}{12}\right) \cdot \frac{144k^2}{144k^2 + \pi^2}.$$

If we integrate from t_0 until t, we get that

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$$\int_{t_0}^{t} e^{k\cdot\tau}\cdot\left(20-10\cdot\cos\left(\frac{2\pi\cdot\tau}{24}\right)\right)\,\mathrm{d}\tau =$$

$$\frac{20e^{k\cdot t}}{k}-10\cdot\left(\frac{e^{k\cdot t}}{k}\cdot\cos\left(\frac{2\pi\cdot t}{24}\right)+\frac{e^{k\cdot t}}{k^2}\cdot\sin\left(\frac{2\pi\cdot t}{24}\right)\cdot\frac{\pi}{12}\right)\cdot\frac{144k^2}{144k^2+\pi^2}-$$
$$\frac{20e^{k\cdot t_0}}{k}-10\cdot\left(\frac{e^{k\cdot t_0}}{k}\cdot\cos\left(\frac{2\pi\cdot t_0}{24}\right)+\frac{e^{k\cdot t_0}}{k^2}\cdot\sin\left(\frac{2\pi\cdot t_0}{24}\right)\cdot\frac{\pi}{12}\right)\cdot\frac{144k^2}{144k^2+\pi^2}.$$

Finally, we get that

$$T(t)=\frac{20}{k}-10\cdot\left(\frac{1}{k}\cdot\cos\left(\frac{2\pi\cdot t}{24}\right)+\frac{1}{k^2}\cdot\sin\left(\frac{2\pi\cdot t}{24}\right)\cdot\frac{\pi}{12}\right)\cdot\frac{144k^2}{144k^2+\pi^2}-$$
$$\frac{20e^{k\cdot(t-t_0)}}{k}-10\cdot\left(\frac{e^{k\cdot(t-t_0)}}{k}\cdot\cos\left(\frac{2\pi\cdot t_0}{24}\right)+\frac{e^{k\cdot(t-t_0)}}{k^2}\cdot\sin\left(\frac{2\pi\cdot t_0}{24}\right)\cdot\frac{\pi}{12}\right)\cdot\frac{144k^2}{144k^2+\pi^2}.$$

Summary

Newton's cooling model was presented in the article. In a simpler case, the ambient temperature is constant. However, when the ambient temperature changes, the model becomes significantly more complicated. Each case is illustrated with an example.

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