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On the Torsional Rigidity of Orthotropic Beams with Rectangular Cross Section

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Abstract. The paper deals with the torsional rigidity of homogenous and orthotropic beam with rectangular cross section. The torsional rigidity of the considered beam is defined in the framework of the Saint-Venant theory of uniform torsion. Exact and approximate solutions are given to the determination of the torsional rigidity. The shape of cross section is determined which gives maximum value of the torsional rigidity for a given cross-sectional area. The dependence of torsional rigidity as a function of the ratio shear moduli of beam is also studied.

Keywords: Orthotropic beam, Saint-Venant torsion, torsional rigidity

Introduction

Figure 1 shows the beam with rectangular cross section which is subjected to torsional load. The material of the beam is elastic, homogenous and Cartesian orthotropic with shear moduli $G_{xz} = G_x$ and $G_{yz} = G_y$. Although the exact solution is known for the twisted orthotropic beam with a rectangular cross section [1,2,3,4], the basic formulae are given for the simplicity, which are directly connected to the torsional stiffness.

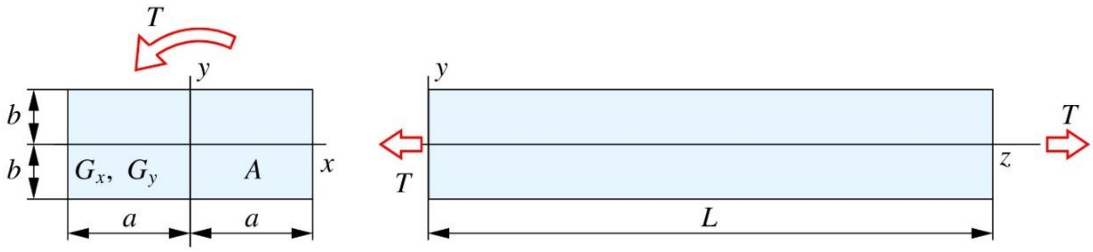

Figure 1 Beam with torsional load.

The Prandtl stress function $U = U(x, y)$ for Cartesian orthotropic beam with solid cross section satisfies the following Dirichlet type boundary-value problem (Figure 1)

$$\frac{1}{G_y}\frac{\partial^2 U}{\partial x^2} + \frac{1}{G_x}\frac{\partial^2 U}{\partial y^2} = -2 \quad (x, y) \in A, \tag{1}$$

$$U(x, y) = 0 \quad (x, y) \in \partial A, \tag{2}$$

where A denotes the cross section and ∂A is the boundary curve of A. In present problem

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$$A=\{(x,y)\mid -a<x<a;\ -b<y<b\}, \tag{3}$$

$$\partial A=\{(x,y)\mid (a^2-x^2)(b^2-y^2)=0;\ |x|\le a;\ |y|\le b\}. \tag{4}$$

The solution of the boundary-value problem for $U=U(x,y)$ is as follows [1,2]

$$U(x,y)=\sum_{k=1}^{\infty}\frac{32G_y a^2}{(2k-1)\pi^3}\left(1-\frac{\cosh\left(\frac{2k-1}{2a}\pi\right)y}{\cosh\left(\frac{2k-1}{2a}\pi b\right)}\right)\sin\left(\frac{(2k-1)\pi}{2a}\right)(x+a),$$

$$-a\le x\le a;\ -b\le y\le b. \tag{5}$$

The expression of the torsional rigidity can be represented as

$$S=2\int_A U\,\mathrm{d}A=\sum_{k=1}^{\infty}\frac{512G_y a^3\left(b-\frac{2a\sqrt{G_y}}{(2k-1)\pi\sqrt{G_x}}\tanh\frac{(2k-1)\pi\sqrt{G_x}b}{2a\sqrt{G_y}}\right)\cos^2 k\pi}{(2k-1)^4\pi^4}, \tag{6}$$

according to [1,2,3,4].

1. Lower bound for the torsional rigidity

Nowinski [5] gave a lower bound formula for the torsional rigidity of homogeneous and Cartesian anisotropic beam. In the present problem the form of this lower bound expression is as follows

$$S>S_L=\frac{\left(\int_A 2\tilde{U}\,\mathrm{d}A\right)^2}{\int_A\left[\frac{1}{G_y}\left(\frac{\partial\tilde{U}}{\partial x}\right)^2+\frac{1}{G_x}\left(\frac{\partial\tilde{U}}{\partial y}\right)^2\right]\mathrm{d}A}, \tag{7}$$

where $\tilde{U}=\tilde{U}(x,y)$ is a statically admissible stress function. $\tilde{U}=\tilde{U}(x,y)$ satisfies the boundary condition

$$\tilde{U}(x,y)=0\quad (x,y)\in\partial A, \tag{8}$$

and it is twice continuously differentiable function of the variables x and y.

Application of formula (7) to the function

$$\tilde{U}(x,y)=(a^2-x^2)(b^2-y^2)\quad (x,y)\in A\cup\partial A \tag{9}$$

gives

$$S_L=\frac{40}{9}\frac{G_x G_y a^3 b^3}{a^2 G_y+b^2 G_x}. \tag{10}$$

2. Upper bound for the torsional rigidity

Nowinski [5] derived an upper bound formula for the torsional rigidity of homogenous and Cartesian anisotropic beam. In the present problem, for Cartesian orthotropic beam this formula gives

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$$S \leq S_U = G_x \int_A y^2 \, \mathrm{d}A + G_y \int_A x^2 \, \mathrm{d}A - \frac{\left(\int_A \left(x G_y \frac{\partial \omega}{\partial y} - y G_x \frac{\partial \omega}{\partial x}\right) \mathrm{d}A\right)^2}{\int_A \left[G_x \left(\frac{\partial \omega}{\partial x}\right)^2 + G_y \left(\frac{\partial \omega}{\partial y}\right)^2\right] \mathrm{d}A}, \tag{11}$$

where $\omega = \omega(x, y)$ is a kinematically admissible torsion function whose second order partial derivatives with respect to x and y are continuous functions.

Substitution for $\omega = \omega(x, y)$ the function

$$\omega(x, y) = xy, \tag{12}$$

into (11) gives

$$S_U = \frac{4 G_x G_y J_x J_y}{G_x J_x + G_y J_y}. \tag{13}$$

For the cross section shown in Figure 1

$$S_U = \frac{48}{9} \frac{G_x G_y a^3 b^3}{a^2 G_y + b^2 G_x}. \tag{14}$$

In formula (14) for rectangular cross section (see Figure 1)

$$J_x = \int_A y^2 \, \mathrm{d}A = \frac{4}{3} a b^3, \quad J_y = \int_A x^2 \, \mathrm{d}A = \frac{4}{3} a^3 b \tag{15}$$

are used.

According to the inequality relation which is valid for harmonic mean and arithmetic mean of two positive real numbers $c = G_x J_x$ and $d = G_y J_y$ we can write

$$\frac{2cd}{c + d} = \frac{4 G_x G_y J_x J_y}{G_x J_x + G_y J_y} \leq \frac{c + d}{2} = \frac{G_x J_x + G_y J_y}{2}, \tag{16}$$

that is

$$S_U = \frac{4 G_x G_y J_x J_y}{G_x J_x + G_y J_y} \leq R_U = G_x J_x + G_y J_y \tag{18}$$

Equality in inequality relation (16) is valid only if

$$G_x J_x = G_y J_y. \tag{19}$$

R_U is a possible upper bound for S, it is weaker than as S_U.

3. The bounding formulae as a function of the ratio of shear moduli

In this section the lower and the upper bounds of torsional rigidity with the exact value of torsional rigidity as a function of ratio of shear moduli $p = G_y / G_x$ is analysed. The following numerical data are used

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$$a = 0.045 \text{ m}, \quad b = 0.025 \text{ m}, \quad G_x = 10 \times 10^{10} \text{ Pa}.$$

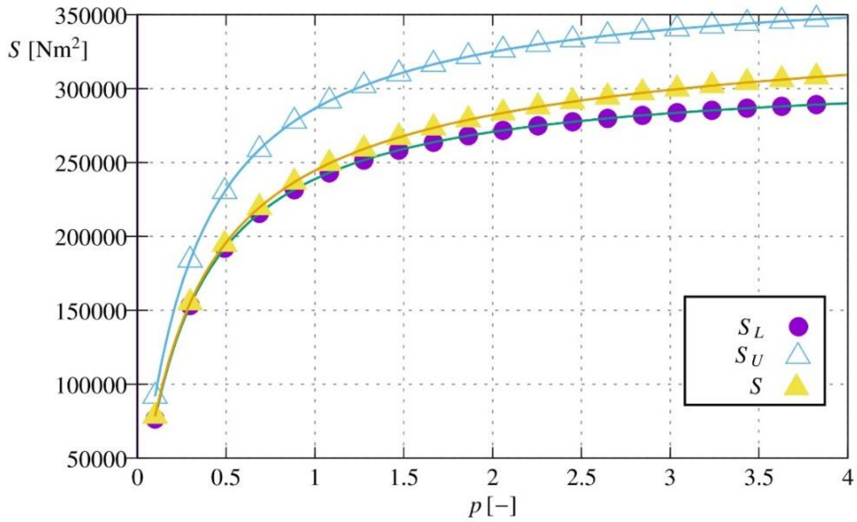

Figure 2 Illustrations of the graphs of function $S_U(p)$, $S(p)$ and $S_L(p)$ for $0.1 \leq p \leq 4$.

4. Determination of the cross section whose torsional rigidity has maximum value for a given cross sectional area

The geometric dimensions of the torsional rigidity with the maximum value for given cross-sectional area is calculated in two steps. Firstly, the approximate value of the geometric dimensions is obtained by the application of lower bound formula (10). Secondly, the obtained value will be made accurate by the application of formula (6). The cross-sectional area in terms of a and b is

$$A = 4ab. \tag{20}$$

Combination of equation (9) with equation (18) gives

$$S_L(a) = \frac{5}{72} \frac{A^3 G_x G_y}{a^2 G_y + \frac{A^2}{16a^2} G_x}. \tag{21}$$

A simple calculation shows that

$$\frac{5}{72} \frac{A^3 G_x G_y}{a^2 G_y + \frac{A^2}{16a^2} G_x} \leq \tilde{S}_0 = \frac{5}{72} \frac{A^3 G_x G_y}{{a_1}^2 G_y + \frac{A^2}{16a_1^2} G_x}, \tag{22}$$

where

$$a_1 = \frac{\sqrt{A}}{2} \sqrt[4]{\frac{G_x}{G_y}}, \tag{23}$$

and

$$b_1 = \frac{A}{4a_1} = \frac{\sqrt{A}}{2} \sqrt[4]{\frac{G_y}{G_x}}. \tag{24}$$

variables a_1 and b_1 can be considered as a first approximation of the geometrical dimension of the optimal cross section. It is very easy to prove that

$$\tilde{S}_0 = \frac{5}{36}\sqrt{G_x G_y A^2} \geq S_L(a). \tag{23}$$

The exact values of cross sectional dimensions of optimal cross section is computed for the following numerical data

$$G_x = 7 \times 10^{10} \text{ Pa}, \quad G_y = 9.5 \times 10^{10} \text{ Pa}, \quad A = 0.0045 \text{ m}^2.$$

For this data the following results are achieved

$$a_1 = 0.021\,064\,329\,660 \text{ m}, \quad b_1 = 0.053\,407\,823\,500 \text{ m}. \tag{24}$$

The plot of function $S = S(a)$ for $0.03 \leq a \leq 0.0315$ is shown in Figure 3. Expression of $S(a)$ is as follows

$$S(a) = \sum_{k=1}^{200} \frac{512 G_y a}{(2k-1)\pi^4}\left(\frac{A}{4a} - \frac{2a\sqrt{G_y}}{(2k-1)\pi\sqrt{G_x}} \tanh\left(\frac{(2k-1)\pi}{8a^2}\sqrt{\frac{G_x}{G_y}}A\right)\right)\cos^2 k\pi. \tag{25}$$

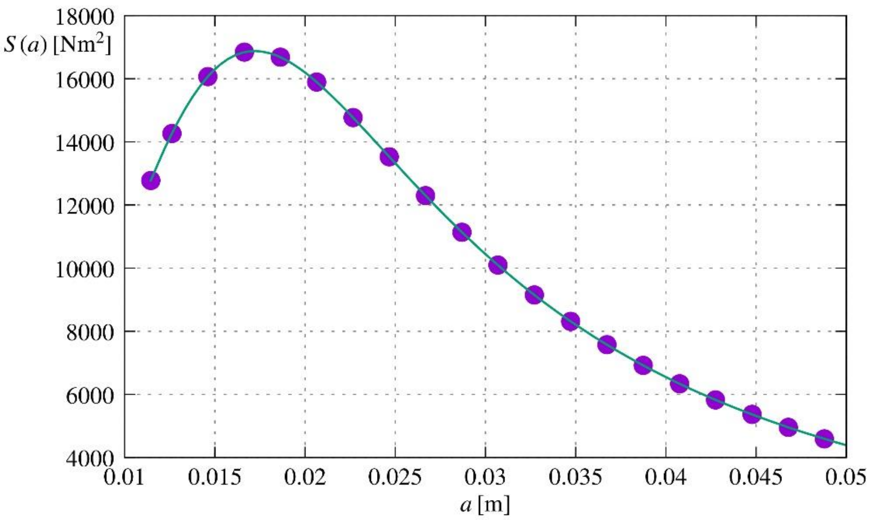

Figure 3 The plot of graph of function $S = S(a)$ for rectangular cross section.

Numerical computations give the results for a_0, b_0 and $S(a_0)$

$$S(a) \leq S(a_0) = 5.052\,363\,2 \times 10^5 \text{ Nm}^2, \tag{26}$$

$$a_0 = 0.021\,064\,330 \text{ m}, \quad b_0 = 0.053\,407\,822\,600 \text{ m}. \tag{27}$$

For $G_x = G_y$ the square cross section gives the maximal value of torsional rigidity for prescribed cross-sectional area A. In this case

$$S_0(a) = \sum_{k=1}^{\infty} \frac{128 a^2}{(2k-1)^5 \pi^5}\left(2\pi A k - A\pi - 8a^2 \tanh\frac{(2k-1)\pi A}{8a^2}\right) \tag{28}$$

and

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$$S_0(a) \leq S_0\left(\frac{\sqrt{A}}{2}\right) \qquad a = b = \sqrt{\frac{A}{2}}. \tag{29}$$

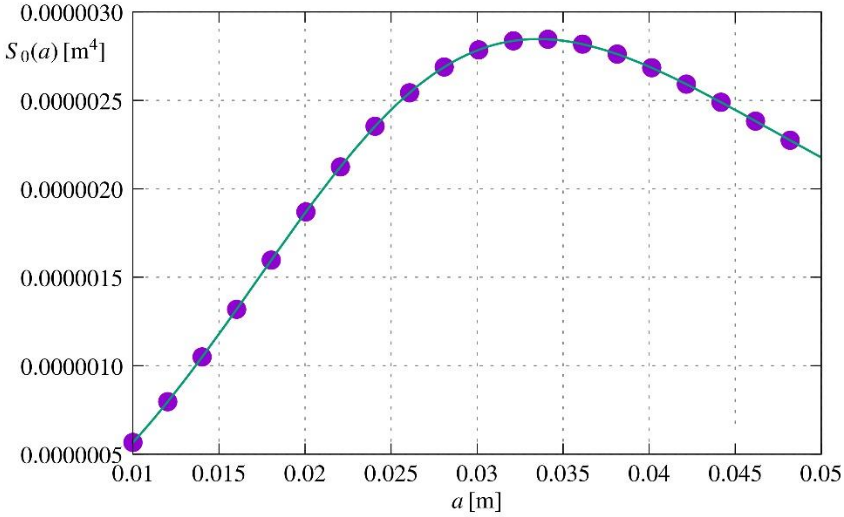

Figure 4 The plot of graph of function $S_0 = S_0(a)$ for square cross section.

The plot of function $S_0(a)$ is shown in Figure 4.

5. Conclusions

Some properties of torsional rigidity of homogenous and Cartesian orthotropic beam are studied. The cross section of the beam is a rectangle. The dimensions of the cross section is determined which gives the maximum value of torsional rigidity for given a cross-sectional area.

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